



On BCK-Topological Modules

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Abstract

In this paper, we introduce the notion of a BCK-topological module in a natural way. It is established that every decreasing sequence of submodules on a BCK-module M over bounded commutative BCK-algebra X is indeed a BCK-topological module. The notion of compatible and strict BCK-module homomorphisms is introduced, and it is furnished that a strict BCK-module homomorphism is an open as well as a continuous mapping. Also, we establish the necessary and sufficient condition for a compatible mapping to be strict.

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1. Introduction

A BCK-module is an action of BCK-algebras on a commutative group. It was introduced by Abujabal, et al. in [1]. They showed in [1] that every bounded implicative BCK-algebra forms a BCK-module over itself and developed the isomorphism theorems. Perveen et. al. introduced the notions of chains, injective and projective on BCK-modules in [13]. Baig and Aslam in [3] introduced the notion of matrices of endomorphisms, topology over decreasing sequence of submodules of a BCK-module, BCK-module over polynomials, Artinian and Noetherian modules and discuss their properties. The theory was further explored in [10] by Kashif and Aslam. They constructed some new examples of BCK-modules in support of the theory, initiated the homology theory of BCK-modules and established that every short exact sequence of complexes admits an exact homology sequence of BCK-modules. In the current work, we further explored the topology introduced by Baig and Aslam in [3] and established that every decreasing sequence of submodules on a BCK-module M over bounded commutative BCK-algebra X is a BCK-topological module. Further, the notions of compact and strict BCK-module homomorphisms are introduced and discussed for various attributes.

Let $(M, +)$ be an abelian group and $(X, *, 0)$ be a BCK-algebra (X -algebra). Then M is called an (left) X -module if there exists a mapping $(x, m) \rightarrow xm$ from $X \times M \rightarrow M$ satisfying the following conditions for all $x, x_1, x_2 \in X$ and $m, m_1, m_2 \in M$; M1) $(x_1 \wedge x_2)m = x_1(x_2m)$, M2) $x(m_1 + m_2) = xm_1 + xm_2$, M3) $0m = 0$, where $x_1 \wedge x_2 = x_2 * (x_2 * x_1)$, M4) $1m = m$, if X is bounded. The set of all (left) X -modules is denoted by X_{mod} .

Let $M \in X_{mod}$, then its additive subgroup N is called X -submodule of M if $N \in X_{mod}$, denoted by $N \leq_X M$. A mapping α from X -module M to another X -module $\bar{M}$ is called an X -homomorphism if for any $x \in X$ and $m_1, m_2 \in M$ the following hold: 1) $\alpha(m_1 + m_2) = \alpha(m_1) + \alpha(m_2)$, 2) $\alpha(xm_1) = x\alpha(m_1)$. An X -homomorphism $f : M_1 \rightarrow M_2$ which is both one to one as well as onto is called an X -isomorphism. The $\ker f$ and f , both in usual sense, are submodules of M_1 and M_2 respectively (see [13]). If $M \in X_{mod}$ and $N \leq_X M$, then quotient group $(M/N, +)$ forms an X -module called the quotient X -module admitted by the scalar multiplication $(a, m + N) \mapsto am + N \forall a \in X, m \in M$ (see for details [1, 3, 13]). Some further extensions of BCK-modules can be seen in [4, 5, 11, 12].

Here, we include some preliminaries from the theory of BCK-algebras. A BCK-algebra is an algebraic system $(X, *, 0)$ that satisfies the following axioms $\forall x, y, z \in X$: BCK1) $(x * y) * (x * z)(z * y) = 0$, BCK2) $(x * (x * y)) * y = 0$, BCK3) $x * x = 0$, BCK4) $0 * x = 0$, BCK5) $x * y = 0, y * x = 0$ implies $x = y$, BCK6) $x * y = 0$ iff $x \leq y$. It can be observed that $(X, \leq)$ forms a poset. In sequel, we denote the BCK-algebra $(X, *, 0)$ simply by X . If X contains an element 1 such that $x \leq 1$ for all x in X , then X is called bounded, X is called commutative if $x \wedge y = y * (y * x)$ holds for all x, y in X , whereas a BCK-algebra X is said to be implicative if $x * (y * x) = x$ for all $x, y \in X$. Throughout the article, the BCK-algebras considered are bounded and commutative, unless stated otherwise. We refer [2, 6, 7, 8, 9] for undefined terms and recent developments regarding BCK-algebras and for topological spaces see [14, 15].

2. BCK-Topological Modules

In this section, the notion of a BCK-topological module is introduced in a canonical way. If M is an X -module and $(M_n | M_n \subseteq M_{n+1})$ is a decreasing sequence of its X -submodules, then we denote it by $(M_n) \in dss(M)$.

Definition 2.1. [3] Let $M \in X_{mod}$ and $(M_n) \in dss(M)$.

Then the collection

$$\mathfrak{S} = \{V \subseteq M | \forall v \in V \exists n \in \mathbb{Z}^+ \text{ such that } v + M_n \subseteq V\} \quad (2.1)$$

of subsets of M forms a topology on M . The topology $\mathfrak{S}$ is referred as a Baig¹ topology on M admitted by $dss(M_n)$. The set of all Baig topologies on an X -module M is denoted by $\mathcal{B}_{top}(M)$.

In [3], the authors proved that every X -submodule N of M is open as well as closed in $\mathfrak{S}$. Also, if a subset A of a topology is both open and closed, then the characteristic function χ_A associated with it is continuous (see [14]). This leads to the following immediate corollary.

Corollary 2.2. Let $M \in X_{mod}$ and $\mathfrak{S}$ be the Baig topology on M admitted by $dss(M_n)$. If $N \leq_X M$, then χ_N is continuous in $\mathfrak{S}$.

Remark 2.3. It is well known that a topological space T is connected if and only if its only sets which are both open and closed are $\emptyset$ and T (see [15]). Also, from [3], every proper X -submodule N of M in $\mathfrak{S}$ is open if and only if it is closed. Therefore, it is concluded that $\mathfrak{S}$ is not connected.

Proposition 2.4. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by the $dss(M_n)$. Then for any m in M the mappings,

- (i) $v : M \rightarrow M$ defined by $v(m) = -m$
- (ii) for $a \in M$, $\tau_a : M \rightarrow M$ defined by $\tau_a(m) = a + m$,

is a homeomorphisms of M onto itself.

¹Imran Baig and M. Aslam defined in [3], the topology on a decreasing sequence of submodules on M .

Proof. (i) It was shown in [3] that $\mathfrak{B} = \{x + M_n : x \in M, n \in \mathbb{Z}^+\}$ forms a base for $\mathfrak{S}$. Indeed for each $U = x + M_n \in \mathfrak{B}$, there exists $V = -x + M_n$ such that $v(V) = v(-x + M_n) = v(-x) + M_n = x + M_n = U$. This shows that v is continuous. Also from the definition, v is one to one, onto and satisfies $v^2(m) = v(-m) = m$ for all $m \in M$ which implies that $v = v^{-1}$. This shows v is also continuous and hence it is a homeomorphism.

(ii) Next, we show that τ_a is a homeomorphism. Let $y = \tau_a(m)$ for some $m \in M$ and $U = y + M_n = \tau_a(m) + M_n \in \mathfrak{B}$ containing y . Then there exists $V = (-a + m) + M_n \in \mathfrak{B}$ such that $m \in V$ and $\tau_a(V) = \tau_a(a + m) = -a + (a + m) + M_n = m + M_n = U$. This shows that τ_a is continuous. It is clear from the definition of τ_a , it is surjective as well as injective. Now, for each $a \in M$, $\tau_{-a}\tau_a(m) = \tau_{-a}(a + m) = -a + a + m = m$ for all $m \in M$. Similarly, $\tau_a\tau_{-a}(m) = m$ for all $m \in M$. Thus $\tau_a\tau_{-a} = \tau_{-a}\tau_a = I \Rightarrow \tau_a^{-1} = \tau_{-a}$. The continuity of τ_a implies the continuity of τ_{-a} . Consequently, τ_a is an homeomorphism. $\square$

Lemma 2.5. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by the dss (M_n) and suppose $M \times M$ be equipped with the product topology. Then the mapping $f : M \times M \rightarrow M$ defined by $f(m, m') = m + m'$ for $m, m' \in M$ is continuous.

Proof. Let $\mathfrak{B} = \{x + M_n : x \in M, n \in \mathbb{Z}^+\}$ be the base for $\mathfrak{S}$ (see [3]) and consider a basic open set $a + M_n \in \mathfrak{B}$ and its inverse image $f^{-1}(a + M_n)$ in $M \times M$. If $f^{-1}(a + M_n) = \emptyset$, then result holds trivially. For nontrivial case, let $f^{-1}(a + M_n) \neq \emptyset$ and $(x, y) \in f^{-1}(a + M_n)$. Therefore, $f(x, y) \in (a + M_n) \Rightarrow f(x, y) + M_n = a + M_n \Rightarrow (x + y) + M_n = a + M_n$. This implies

$$f^{-1}(a + M_n) = f^{-1}((x + y) + M_n).$$

It is easy to see that, for basic open sets $(x + M_n), (y + M_n) \in \mathfrak{B}$

$$(x + M_n) \times (y + M_n) \subseteq f^{-1}(x + y + M_n)$$

This implies $(x, y) \in f^{-1}(a + M_n)$. Hence $f^{-1}(a + M_n)$ is open. Consequently, f is continuous. This completes the proof. $\square$

Lemma 2.6. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by the dss (M_n). If for each $x \in X$, $\mu_x : M \rightarrow M$ is a mapping defined by

$$\mu_x(m) = xm. \quad (2.2)$$

Then μ_x is continuous.

Proof. Let $y = \mu_x(m) \in M$; $m \in M$ and $U = y + M_n \in \mathfrak{B}$ be a basic open set containing y . Then there exists $V \in \mathfrak{B}$ containing m and satisfying $\mu_x(V) \subseteq U$. Indeed, for $U = y + M_n = xm + M_n \in \mathfrak{B}$ there exists $V = m + M_n \in \mathfrak{B}$ such that $\mu_x(V) = \mu_x(m + M_n)$.

Now for $a \in \mu_x(m + M_n)$ it is easy to see that $a \in xm + M_n = \mu_x(m) + M_n$. This implies $\mu_x(m + M_n) \subseteq \mu_x(m) + M_n$.

Similarly, $\mu_x(m) + M_n \subseteq \mu_x(m + M_n)$. This implies $\mu_x(m) + M_n = \mu_x(m + M_n)$. Therefore, $\mu_x(V) = xm + M_n \in \mathfrak{B}$ and hence $\mu_x(V) \subseteq U$.

This shows μ_x is continuous. $\square$

Remark 2.7. The continuity of the operations of addition $+$ and the X -multiplication in M , with respect to the $\mathfrak{S}$, in Lemma 2.5 and 2.6, motivates us to introduce the following notion.

Definition 2.8. An X -module M is said to be a BCK-topological module, if the operations of addition $+$ and the X -multiplication in M are continuous with respect to certain topology on M .

Theorem 2.9. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by the dss (M_n). Then M is an X -topological module.

Proof. Clear from above definition and Lemma 2.5-2.6. $\square$

In the following theorem, the continuity of an X -homomorphism between X -modules is characterised.

Theorem 2.10. Let $M, M' \in X_{mod}$ and $\mathfrak{S}, \mathfrak{S}'$ be their respective Baig topologies admitted by $dss(M_n)$ and $dss(M'_n)$ respectively. If $f : M \rightarrow M'$ is an X -homomorphism, Then an X -homomorphism $f : M \rightarrow M'$ is continuous iff it is continuous at $0 \in M$.

Proof. The continuity of f on M trivially implies that it is continuous at $0 \in M$. Conversely, let f be continuous at 0, it will be shown that it is continuous at each $m \in M$.

Since f is continuous at $0 \in M$, therefore for any basic open set $M'_k \subseteq M'$ such that $f(0) \in M'_k$, there exists a basic open set $M_n \in M'$ such that $0 \in M_n$ and

$$f(M_n) \subseteq M'_k. \quad (2.3)$$

Let $y = f(m)$ be an arbitrary element in M'_n and $U = y + M'_k$ be a basic open set containing y .

Then there exists $V = m + M_n \in \mathfrak{B}$, such that the X -homomorphism of f and Eq. 2.3 implies $f(V) = f(m) + f(M_n) \subseteq f(m) + M'_k = U$. Shows $f(V) \subseteq U$ and hence f is continuous. $\square$

3. Compatible and Strict X -Module Homomorphisms

In the following section, the notions of compatible and strict X -module homomorphisms is introduced. It will be explored for various features and it will be will show that every strict mapping is open as well as continuous. The notions of quotient Baig topology and induced Baig topology will be introduced and explored. Finally, the necessary and sufficient condition for a compatible X -module homomorphisms to be strict, will be established.

Definition 3.1. Let M and M' be X -modules and $\mathfrak{S}$ and $\mathfrak{S}'$ be the respective Baig topologies on them admitted by $dss\{M_n | M_n \subseteq M_{n+1}\}_{n \in \mathbb{Z}^+}$ and $\{M'_n | M'_n \subseteq M'_{n+1}\}_{n \in \mathbb{Z}^+}$ respectively. Then an X -homomorphisms $f : M \rightarrow M'$ is said to be **compatible** with the Baig topologies if

$$f(M_n) \subseteq M'_n \quad \forall n \in \mathbb{Z}^+ \quad (3.1)$$

and f is said to be **strict** if

$$f(M_n) = f(M) \cap M'_n \quad \forall n \in \mathbb{Z}^+ \quad (3.2)$$

Proposition 3.2. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be Baig topologies on X -modules M and M' admitted by $dss(M_n)$ and $dss(M'_n)$ respectively. Then an X -homomorphisms $f : M \rightarrow M'$ is compatible if it is strict.

Proof. Let $f : M \rightarrow M'$ be strict. Then Eq. (3.2) implies $f(M_n) = f(M) \cap M'_n \subseteq M'_n \quad \forall n \in \mathbb{Z}^+$. Shows that f is compatible. $\square$

Proposition 3.3. Let $M, M' \in X_{mod}$ and $f : M \rightarrow M'$ be an X -homomorphism. If $K \leq_X M$, then

$$f(K + m) = f(K) + f(m) \quad \forall m \in M$$

Proof. Let $m' \in f(K + m) \Rightarrow m' = f(k + m); k \in K \Rightarrow m' = f(k) + f(m) \Rightarrow m' \in f(K) + f(m) \Rightarrow f(K + m) \subseteq f(K) + f(m)$.

For reverse inclusion let $m' \in f(K) + f(m) \Rightarrow m' = f(k) + f(m); k \in K, \Rightarrow m' = f(k + m) \Rightarrow m' \in f(K + m) \Rightarrow f(K) + f(m) \subseteq f(K + m)$

Hence $f(K + m) = f(K) + f(m) \quad \forall m \in M$. $\square$

Proposition 3.4. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be in $\mathcal{B}_{top}(M)$ and $\mathcal{B}_{top}(M')$ admitted by $dss(M_n)$ and $dss(M'_n)$ respectively. Then every strict X –homomorphisms $f : M \rightarrow M'$ is an open mapping.

Proof. It suffices to show that for an arbitrary basic open set $m + M_n$ in M , $f(m + M_n)$ is an open set. From Proposition 3.3 and the strictness of f , one arrives at $f(M_n + m) = (f(M) \cap M'_n) + f(m)$. Next we show that

$$(f(M) \cap M'_n) + f(m) = f(M) \cap (M' + f(m)).$$

Let $y \in (f(M) \cap M'_n) + f(m)$. Then there exists $m_1 \in M$ such that $y = f(m_1) + f(m) = f(m_1 + m)$. This implies $y \in f(M)$. Also it is evident from above that $y \in M'_n + f(m)$. This implies $y \in f(M) \cap (M'_n + f(m))$ and hence $(f(M) \cap M'_n) + f(m) \subseteq f(M) \cap (M'_n + f(m))$.

The reverse inclusion can be proved in a similar manner. Hence

$$(f(M) \cap M'_n) + f(m) = f(M) \cap (M'_n + f(m))$$

Now, $f(M)$ is open in M' , since $f^{-1}(f(M)) = M$. Also $M'_n + f(m)$ is basic open set in M' , therefore, $f(M) \cap (M'_n + f(m))$ is an open set being intersection of open sets.

Consequently, $f(M_n + m)$ is open. This completes the proof. $\square$

Proposition 3.5. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be Baig topologies on X –modules M and M' admitted by $dss \{M_n | M_n \subseteq M_{n+1}\}_{n \in \mathbb{Z}^+}$ and $\{M'_n | M'_n \subseteq M'_{n+1}\}_{n \in \mathbb{Z}^+}$ respectively. If $f : M \rightarrow M'$ is compatible with Baig topologies $\mathfrak{S}$ and $\mathfrak{S}'$, then it is continuous everywhere.

Proof. The compatibility of $f : M \rightarrow M'$ implies $f(M_n) \subseteq M'_n \forall n \in \mathbb{Z}^+$. In order to show that f is continuous everywhere, it is sufficient to show its continuity at zero.

Let M'_n be an arbitrary basic open set containing $f(0)$. Then using compatibility of f , it follows that $0 \in M_n \subseteq f^{-1}(M'_n)$. This shows f is continuous at 0 and hence by Proposition 2.10 f is continuous everywhere. This completes the proof. $\square$

The following are immediate corollaries of the ongoing discussions.

Corollary 3.6. Every strict mapping is continuous.

Proof. The following diagram explains the proof.

$$\text{Strictness} \xrightarrow{\text{Proposition 3.2}} \text{Compatiblity} \xrightarrow{\text{Proposition 3.5}} \text{Continuity}$$

$\square$

Corollary 3.7. Every strict X –module isomorphism $f : M \rightarrow M'$ is an homeomorphism..

Proof. Let f be a strict X –module isomorphism. Then by Corollary 3.6 and Proposition 3.4, it follows that f is an open mapping. Since f is a bijection such that f and f^{-1} , therefore, f is a homeomorphism. $\square$

Proposition 3.8. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by its $dss(M_n)$. Then for any submodule K of M the sequence

$$\{K_n | K_n = K \cap M_n\}_{n \in \mathbb{Z}^+} \quad (3.3)$$

admits a topology on K .

Proof. Since K and M_n are both submodules of $M \forall n$, therefore, $K_n = K \cap M_n \forall n$ is also a submodule of M such that $K_n \supseteq K_{n+1} \forall n$. Indeed, the set $\{K_n | K_n = K \cap M_n\}_{n \in \mathbb{Z}^+}$ forms a *dss* of K . Let $\mathfrak{S}_i$ be a collection of subsets of K defined as

$$\mathfrak{S}_i = \{U \subseteq K | u \in U \exists n \in \mathbb{Z}^+ \text{ such that } u + K_n \subseteq U\}. \quad (3.4)$$

Then by Definition 2.1 $\mathfrak{S}_i$ forms a Baig topology on K . □

The topology $\mathfrak{S}_i$ introduced in Eq. (3.4) is referred as Induced Baig topology on K . This leads to the following definition.

Definition 3.9. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by *dss* (M_n) . Then for each $K \leq_X M$ with respective *dss* $\{K_n | K_n = K \cap M_n\}_{n \in \mathbb{Z}^+}$, the collection $\mathfrak{S}_i = \{U \subseteq K | \forall u \in U \exists n \in \mathbb{Z}^+ \text{ such that } u + K_n \subseteq U\}$ of subsets of K forms a topology on K called the induced Baig topology on K .

The following proposition shows that the induced Baig topology on K , is a relative topology on K .

Proposition 3.10. Let $M \in X_{mod}$ and $\mathfrak{S} \in \mathcal{B}_{top}(M)$ admitted by *dss* (M_n) . Then for any $K \leq_X M$, the topology $\mathfrak{S}_i$, is a relative topology on K w.r.t. $\mathfrak{S}$.

Proof. Let $l \in K$ be an arbitrary element and consider $y \in l + K \cap M_n$. Then there exists some $m \in K \cap M_n$ such $y = l + m$. This implies $y \in K$. Also $m \in M_n$ implies $y = l + m \in l + M_n$. This shows that $y \in (l + M_n) \cap K$ and hence $l + K \cap M_n \subseteq (l + M_n) \cap K$. The reverse inclusion can be proved similarly. Therefore, one obtains the following:

$$(l + M_n) \cap K = l + K \cap M_n = l + K_n \quad \because K_i = l + K_i \in \mathfrak{S}_i$$

Since $l + M_n$ is an open set in $\mathfrak{S}_i$, therefore $(l + M_n) \cap K$ is open in relative topology on K . Therefore, basic open sets of $\mathfrak{S}_i$ are open set of the relative topology on K . Consequently, $\mathfrak{S}_i$ is a relative topology on K w.r.t. the Baig topology $\mathfrak{S}$. □

Proposition 3.11. Let $\mathfrak{S}$ be a Baig topology on an X -module M admitted by its *dss* $\{M_n | M_n \subseteq M_{n+1}\}_{n \in \mathbb{Z}^+}$. Then for any submodule K of M the sequence

$$\left\{ \frac{M_n + K}{K} | n \in \mathbb{Z}^+ \right\} \quad (3.5)$$

admits a topology on the factor X -module $\frac{M}{K}$.

Proof. Since $M_n + K$ is a submodule of M , therefore, $\frac{M_n + K}{K}$ is a submodule of the factor module $\frac{M}{K}$. Also $M_n \supseteq M_{n+1} \forall n \in \mathbb{Z}^+$, therefore, $M_n + K \supseteq M_{n+1} + K \forall n \in \mathbb{Z}^+$. This implies $\frac{M_n + K}{K} \supseteq \frac{M_{n+1} + K}{K} \forall n \in \mathbb{Z}^+$. Therefore, $\left\{ \frac{M_n + K}{K} | n \in \mathbb{Z}^+ \right\}$ forms a *dss* of the factor X -module $\frac{M}{K}$. Let $\mathfrak{S}_f$ defines a collection of subsets of $\frac{M}{K}$ as below:

$$\mathfrak{S}_f = \left\{ \frac{N}{K} \subseteq \frac{M}{K} \mid \forall \bar{U} \in \frac{N}{K} \exists n \in \mathbb{Z}^+ \text{ such that } \bar{U} + \frac{M_n + K}{K} \subseteq \frac{N}{K} \right\}. \quad (3.6)$$

Then by Definition 2.1 $\mathfrak{S}_f$ forms a topology on $\frac{M}{K}$. □

The above proposition leads to the following definition.

Definition 3.12. Let M be an X -module and $\mathfrak{S}$ be a Baig topology on M admitted by its dss $\{M_n | M_n \subseteq M_{n+1}\}_{n \in \mathbb{Z}^+}$. Then for any submodule K of M and the dss $\{\frac{M_n+K}{K} | n \in \mathbb{Z}^+\}$ of its factor module $\frac{M}{K}$, the collection of subsets $\mathfrak{S}_f = \{U \subseteq K | \forall u \in U \exists n \in \mathbb{Z}^+ \text{ such that } u + K_n \subseteq U\}$ admits a topology on $\frac{M}{K}$ called the factor Baig topology on M w.r.t. K .

Proposition 3.13. Let M be an X -module. Then for any submodule K and respective factor X -module M/K following hold:

a) The sequence

$$K \xrightarrow{i} M \xrightarrow{f} M/K \quad (3.7)$$

where $i(k) = k$; $k \in K$ is the inclusion mapping and $f(m) = m + K$; $m \in M$ is the natural X -homomorphism, is an exact sequence.

b) The inclusion mapping i and the natural X -homomorphism f are strict if K, M and M/K are equipped with their respective Baig topologies $\mathfrak{S}_i, \mathfrak{S}$ and $\mathfrak{S}_f$ (c.f. eqs. (2.1,3.4,3.6)).

Proof. **a):** The inclusion mapping i and the mapping f are indeed natural homomorphism such that $i = K = \ker f$. Therefore, diagram (3.7) forms an exact sequence.

b): Since $K_n = K \cap M_n$ by Definition 3.9, therefore, $i(K_n) = i(K \cap M_n) = K \cap M_n = i(K) \cap M_n$. Hence by Eq. (3.2), i is strict.

Next, we will show that f is strict. Since $f(M) = \frac{M}{K}$ and the fact that $M_n + K \subseteq M$ therefore, $f(M) \cap \frac{M_n+K}{K} = \frac{M_n+K}{K}$. Also it is easy to see that $f(M_n) = \frac{M_n+K}{K}$. Hence $f(M) \cap \frac{M_n+K}{K} = f(M_n)$. Therefore by Eq. (3.2), f is strict. □

Corollary 3.14. i and f are open mappings.

Proof. An immediate implication of Proposition 3.4. □

Theorem 3.15. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be Baig topologies on X -modules M and M' admitted by dss $\{M_n | M_n \subseteq M_{n+1}\}_{n \in \mathbb{Z}^+}$ and $\{M'_n | M'_n \subseteq M'_{n+1}\}_{n \in \mathbb{Z}^+}$ respectively. If $f : M \rightarrow M'$ is a compatible X -homomorphism, then for all n , f induces an X -homomorphism $f_n : \frac{M}{M_n} \rightarrow \frac{M'}{M'_n}$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \downarrow \phi_n & & \downarrow \phi'_n \\ \frac{M}{M_n} & \xrightarrow{f_n} & \frac{M'}{M'_n} \end{array} \quad (3.8)$$

commutes, where ϕ_n and ϕ'_n are natural X -homomorphisms from M onto M/M_n and M' onto M'/M'_n respectively.

Proof. Let $f_n : \frac{M}{M_n} \rightarrow \frac{M'}{M'_n}$ be defined in a natural way as $f_n(m + M_n) = f(m) + M'_n$; $m \in M$. First we will establish that f_n is well defined.

Let $\bar{m}_1 = m_1 + M_n$, $\bar{m}_2 = m_2 + M_n \in \frac{M}{M_n}$ such that $\bar{m}_1 = \bar{m}_2$. Then $m_1 - m_2 \in M_n$ and therefore, $f(m_1 - m_2) \in f(M_n) \subseteq M'_n$ since f is compatible. The X -homomorphism of f implies that $f(m_1) + M' = f(m_2) + M'$ and hence $f_n(\bar{m}_1) = f_n(\bar{m}_2)$. This shows that f is well defined. Also the natural induction of f_n from the X -homomorphism f implies that it is also an X -homomorphism.

The only thing remains to establish is that diagram (3.8) commutes, i.e., $\phi'_n f = f_n \phi_n$. Let $m \in M$. Then $f_n \phi_n(m) = f_n(\phi_n(m))$. Also $\phi_n(m) = m + M_n$ implies that $f_n \phi_n(m) = f_n(m + M_n) = f(m) + M'_n = \phi'_n(f(m)) = \phi'_n f(m) \quad \forall m \in M$. Hence $f_n \phi_n = \phi'_n f$. This completes the proof. □

Corollary 3.16. Consider the notation used in the Theorem 3.15 and the diagram (3.8). Then for each $n \in \mathbb{Z}^+$, there exists an X –homomorphism $\alpha_n : \text{Ker } f \rightarrow \text{Ker } f_n$.

Proof. Let $k \in \text{ker } f$. Then $f(k) = 0 \in M'_n \forall n$ since M'_n are submodules of M' for all n . This implies $\phi'_n(f(k)) = \phi'_n f(k) = M'_n$ is the zero of $\frac{M'}{M_n}$. Since the diagram (3.8) is commutative, therefore, $f_n \phi_n(k) = \phi'_n f(k) = M'_n$. This implies that $\phi_n(k) \in \text{ker } f_n$. Let $\alpha_n : \text{ker } f \rightarrow \text{ker } f_n$ be defined as:

$$\alpha_n(k) = \phi_n(k)$$

Indeed, α_n is a well defined X –homomorphism, since ϕ_n is a well defined X –hom-omorphism. This completes the proof. $\square$

The following theorem presents the necessary and sufficient condition for a compatible mapping to be strict.

Theorem 3.17. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be Baig topologies on X –modules M and M' admitted by $dss \{M_n | M_n \subseteq M_{n+1}\}_{n \in \mathbb{Z}^+}$ and $\{M'_n | M'_n \subseteq M'_{n+1}\}_{n \in \mathbb{Z}^+}$ respectively. Then a compatible X –homomorphism $f : M \rightarrow M'$ is strict if and only if for all $n \in \mathbb{Z}^+$ the associated mapping $\alpha_n : \text{ker } f \rightarrow \text{ker } f_n$ is an X –epimorphism.

Proof. Let $f : M \rightarrow M'$ be strict. Then $f(M_n) = f(M) \cap M'_n \forall n \in \mathbb{Z}^+$. We will show that α_n is an epimorphism. Let $k' \in \text{ker } f_n$. Then $f_n(k') = 0 ; k' \in M$. Since ϕ_n is surjective, therefore, there exists some $k \in M$ such that $f_n \phi_n(k) = 0$. Therefore, from Theorem 3.15 $\phi'_n f(k) = M'_n$ and hence $f(k) \in \text{ker } \phi'_n = M'_n$. This implies $f(k) \in f(M) \cap M'_n = f(M_n)$. Implies there exists $m_n \in M_n$ such that $f(k) = f(m_n)$. Therefore, $k - m_n \in \text{Ker } f$ and from Corollary 3.16, simple calculations reveal that $\phi_n(k - m_n) = \phi_n(k) = k'$. Since k' was arbitrary, therefore, α_n is an X –epimorphism.

Conversely, assume that $\alpha_n : \text{ker } f \rightarrow \text{ker } f_n$ be an X –epimorphism, we will show that f is strict. Indeed, the compatibility of f implies $f(M_n) \subseteq f(M) \cap M'_n$. For reverse inclusion, let $x' \in f(M) \cap M'_n$. Then there exists $x \in M$ such that $f(x) = x'$. Now the commutativity of diagram 3.8 implies that $\phi_n(x) \in \text{Ker } f_n$. Since $\alpha_n : \text{ker } f \rightarrow \text{ker } f_n$ is an X –epimorphism, therefore, there exists $y \in \text{ker } f$ such that $\alpha_n(y) = \phi_n(x)$. Now from Corollary 3.16, it implies that $\alpha_n(y) = \phi_n(y)$. This further implies that $x - y \in \text{ker } \phi_n = M_n$. Also $f(x - y) = f(x) - f(y) = f(x) = x' \because y \in \text{ker } f$. This implies $f(x - y) = x' \in f(M_n) \because x - y \in M_n$. Hence, $f(M) \cap M'_n \subseteq f(M_n)$ Consequently, $f(M) \cap M'_n = f(M_n)$ by Definition 3.1. This completes the proof. $\square$

Corollary 3.18. If in the last proposition $\alpha_n : \text{ker } f \rightarrow \text{ker } f_n$ is an X –epimorphism, then f is a continuous as well as an open mapping.

Proof. Indeed, in this case Theorem 3.17 implies f is strict and therefore, from Corollary 3.6 and Proposition 3.4, it follows that f is continuous and open respectively. $\square$

4. A direction to explore for BCK-modules

The theory developed above can be further explored in different directions. One such direction is to discuss the completeness of BCK-module w.r.t. the Baig Topology. Some interesting notions and results have already been established regarding the necessary and sufficient condition of completeness of a BCK-modules. These results will be communicated after some further refinement and addition in near future.

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