

Analytical solution of fractional partial differential equations for the fluid flow over an infinite horizontal plate

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Abstract

The present study deals with the unsteady, incompressible viscous fluid flow with hybrid fractional derivative namely constant proportional Caputo type with singular kernel. In this paper, we find new analytical solutions of a well-known problem of the fluid dynamics known as Stokes' first problem. Using dimensional analysis governing equation convert into non-dimensional form and then extended with hybrid fractional operator of singular kernel and series solutions are obtained by Laplace transform method. Some graphs are captured through software for fractional parameter α , t and Reynolds number Re respectively. As a result, we found that hybrid velocity shows dual solutions for small and large values of time for different values of fractional parameter α . Further, we compared the present result with all the existing fractional operators of kernel power law, exponential law and Mittag-Leffler law. Momentum boundary thickness increases for increasing time while reduces by increasing the value of α and this fact be seen in comparison of all operators. Further, observed that constant proportional Caputo type operator is well suited in exhibiting the decay of velocity than all the existing fractional operators with power law (C), exponential law (CF) and Mittag-Leffler law (ABC).

Keywords: Constant proportional fractional derivative; New analytical solution; Stokes' problem

Mathematics Subject Classification: 2000 : 06F35

1. Introduction

Khalil et al. [1] introduced the formal definition of local conformable fractional derivative in (2014). These conformable fractional operators lack some properties which were addressed in [2]. Abdeljawad et al. [3] discussed the conformable calculus with Taylor power series, chain rule, integration by parts and so on. Zhao et al. [4] gives the physical interpretation of general conformable derivative. In (2015) Benkhettou et al. [5] introduces the definition of conformable calculus for any arbitrary time scale. Rahmat et al. [6] introduced the new definition of conformable derivative of arbitrary time scale in (2019). Tasbozan et al. [7] found new analytical solutions of fractional partial differential equations with conformable derivative. Salahshour et al. [8] applied new conformable fractional derivative under uncertainty. Some more new definitions of conformable fractional derivatives with their applications in fractional PDE's and control theory can be seen in the references [9-12].

Fractional operators are usually used to see the history or memory of the functions based on different kernels like, power law kernel, exponential kernel, Mittag-Leffler kernel etc. These kernels are usually

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appear in many physical processes like decay phenomena. They are more appropriate in dealing with complex phenomena in real world problems. So, during the last five years many researchers successfully applied these fractional operators in fluid dynamics, heat transfer, nanotechnology, epidemic model, Chaotic, Ebola diseases, in telegraph equation and many more with power law (C), exponential law (CF) and Mittag-Leffler law (ABC) fractional derivatives and obtained interesting response. Some applications can be seen in the references for instance [14-35].

In [36] Baleanu introduces a new kind of fractional operator with power law and called it a hybrid fractional operator with non-local and singular kernel. Hybrid fractional operator is combination of two operators namely; proportional or conformable and Caputo fractional derivatives. They combined both the operators into a single operator which also known as constant proportional Caputo type fractional derivative. In this paper they solved the fractional partial differential equations with Laplace transform method and expressed the solutions in terms of bivariate Mittag-Leffler functions. This operator has application in fractional partial differential equation and control theory. Constant proportional Caputo type derivative is defined as [36]

$${}^{PC}C_t^\alpha h(t) = [\Gamma(1-\alpha)]^{-1} \int_0^t \frac{(L_1(\alpha, \tau)h(\tau) + L_o(\alpha, \tau)\dot{h}(\tau))}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1. \quad (1.1)$$

In the special case if L_1 and L_o are independent of time then constant proportional Caputo is defined as [36]

$${}^{CPC}C_t^\alpha h(t) = [\Gamma(1-\alpha)]^{-1} \int_0^t \frac{(L_1(\alpha)h(\tau) + L_o(\alpha)\dot{h}(\tau))}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1. \quad (1.2)$$

with limiting cases as follows

$$\lim_{\alpha \rightarrow 0} {}^{PC}C_t^\alpha h(t) = \lim_{\alpha \rightarrow 0} {}^{CPC}C_t^\alpha h(t) = \int_0^t h(\tau) d\tau, \quad (1.3)$$

$$\lim_{\alpha \rightarrow 1} {}^{PC}C_t^\alpha h(t) = \lim_{\alpha \rightarrow 1} {}^{CPC}C_t^\alpha h(t) = \dot{h}(t). \quad (1.4)$$

The Laplace transform of constant proportional Caputo is given as [36]

$$\mathcal{L}\left\{{}^{CPC}C_t^\alpha h(t)\right\} = \left\{\frac{L_1(\alpha)}{s} + L_o(\alpha)\right\} s^\alpha - L_o(\alpha) s^{\alpha-1} h(0). \quad (1.5)$$

For the moment, there is no single result in the literature discussed with this new operator. Therefore, our interest here to apply the most recent fractional operator namely hybrid fractional operator to the most fundamental problem of fluid dynamics which is known as Stokes' first problem. Solution obtained via Laplace transform in terms of series and compared with the solutions of all the existing fractional operators. In this paper, we show that which operator is better in exhibiting decay of the velocity of the fluid. We have plotted some graphs for fractional parameters for small and large time. Also, drawn some comparison between present and classical, power law (C), exponential law (CF) and Mittag-Leffler law (ABC) and fractal and fractional derivatives and presented in graphically.

2. Mathematical modeling with hybrid fractional operator and its solution

Consider the unsteady flow of an incompressible viscous fluid over a flat (solid) plate which fills the half space $y > 0$ occupying xz -plane. In the beginning, both fluid and plate are assumed to be at rest. After some passed, plate starts to move with a constant velocity U and the flow occur along x -axis

figure 1. Its further assumed that there is no body force, no pressure gradient in the flow direction and incompressibility conditions satisfy for the velocity field of the form $V = (u(y, t), 0, 0)$.

The equation governing for $u(y, t)$, is obtained from law of conservation of momentum and is given by

$$\rho u_t(y, t) = \mu u_{yy}(y, t), \quad (2.1)$$

with set of initial and boundary conditions are:

$$u(y, 0) = 0 \quad \forall y, \quad u(0, t) = UH(t), \quad \text{for } t > 0, \quad u(y, t) \rightarrow 0, \quad \text{as } y \rightarrow \infty. \quad (2.2)$$

Introducing the non-dimensional variables,

$$u^* = \frac{u}{U}, \quad y^* = \frac{y}{L}, \quad t^* = \frac{U}{L}t,$$

equation (1) and after ignoring the *, we get,

$$\text{Re} u_t(y, t) = u_{yy}(y, t), \quad (2.3)$$

with set of conditions

$$u(y, 0) = 0, \quad u(0, t) = H(t), \quad u(y, t) \rightarrow 0, \quad \text{as } y \rightarrow \infty. \quad (2.4)$$

where $\text{Re} = \frac{\rho UL}{\mu}$ is Reynolds number and is defined as a dimensionless number which is defined as the ratio of inertia force by viscous force is known as Reynolds number and is used to characterized the flow of fluid.

By applying Eq. (2) to Eq. (8), we get CPC fractional model

$$\text{Re}^{CPC} C_t^\alpha u(y, t) = u_{yy}(y, t), \quad 0 < \alpha < 1. \quad (2.5)$$

By taking Laplace transform of Eq. (9) and (10) we have

$$\bar{u}_{yy}(y, s) = \text{Re} L_o(\alpha) \left[1 + \frac{L_1(\alpha)}{L_o(\alpha)} s^{-1} \right] s^\alpha \bar{u}(y, s), \quad (2.6)$$

initial and boundary conditions:

$$\bar{u}(0, s) = \frac{1}{s}, \quad \bar{u}(y, s) \rightarrow 0, \quad \text{as } y \rightarrow \infty. \quad (2.7)$$

The solution of Eq. (11) subject Eq. (12) is given by or

$$\bar{u}(y, s) = \frac{1}{s} e^{-y \sqrt{\text{Re} \left[\frac{L_1(\alpha)}{s} + L_o(\alpha) \right] s^\alpha}} \quad (2.8)$$

Eq. (13) can be written as in more suitable form so that we can find inverse Laplace transform analytically,

$$\bar{u}(y, s) = \frac{1}{s} + \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-y \sqrt{\text{Re}})^p (L_o(\alpha))^{\frac{p}{2}-k} (L_1(\alpha))^k}{p! k! s^{1-\frac{\alpha p}{2}+k}} \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)} \quad (2.9)$$

Taking Laplace inverse of Eq. (14) we get the final expression for solution in terms of series

$$u(y, t) = 1 + \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-y \sqrt{\text{Re}})^p (L_o(\alpha))^{\frac{p}{2}-k} (L_1(\alpha))^k t^{-\frac{\alpha p}{2}+k}}{p! k! \Gamma(1-\frac{\alpha p}{2}+k)} \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+1-k)} \quad (2.10)$$

Remark: The series we have found in the t-domain will be convergent everywhere.

Now, we find the solution of the problem given in Eqs. (8) and (9), with different fractional operators by applying the Laplace transform method one by one which are listed below.

- Velocity field with power law kernel C.

Solution of Eq. (8) with initial and boundary conditions given in Eq. (9) with Caputo fractional derivative and Laplace transform is given below

$$u(y,t) = 1 + \sum_{p=1}^{\infty} \frac{(-y\sqrt{\text{Re}})^p t^{-\frac{\alpha p}{2}}}{p! \Gamma(1 - \frac{\alpha p}{2})} \quad (2.11)$$

- Velocity field with exponential kernel CF.

Solution of Eq. (8) with initial and boundary conditions given in Eq. (9) with CF fractional derivative and Laplace transform is given below

$$u(y,t) = 1 + \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-y\sqrt{\text{Re}})^p (-\alpha)^k t^k}{p! k! (1 - \alpha)^{\frac{p}{2} + k}} \frac{\Gamma(\frac{p}{2} + k)}{\Gamma(k+1) \Gamma(\frac{p}{2})} \quad (2.12)$$

- Velocity field with Mittag-Leffler kernel ABC.

Solution of Eq. (8) with initial and boundary conditions given in Eq. (9) with ABC fractional derivative and Laplace transform is given below

$$u(y,t) = 1 + \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-y\sqrt{\text{Re}})^p (-\alpha)^k t^{\alpha k}}{p! k! (1 - \alpha)^{\frac{p}{2} + k}} \frac{\Gamma(\frac{p}{2} + k)}{\Gamma(\alpha k + 1) \Gamma(\frac{p}{2})} \quad (2.13)$$

3. Numerical results and discussion

Novel fractional operator called hybrid fractional operator applied in the fundamental problem of fluid dynamics. This new operator is a combination of constant proportional and Caputo type with singular kernel. Obtained solutions are expressed in terms of series. We have drawn some comparison with existing fractional derivatives and some graphs are presented graphically.

Figure 2 and 3 are depicted to see the impact of Re on the innovative fractional derivative and all other existing derivatives. It is found that by increasing the values of Re the fluid velocity is also decreases for all fractional parameters. This happened due to the fact that Re is a dimensionless number usually appear in fluid dynamics which characterized the flow behavior. It is ratio between inertial force and viscous force. It is the relative strength of inertial forces to viscous forces. The relative strength of these two actions their ratio does have a lot of influence on how the fluid flow behaves. Therefore, viscous are more dominant in this case and responsible to slow down the fluid flow as well as reduce the boundary layer thickness between the models. Figure 4 is depicted for large values of time $t=5$, $t=10$ and $t=15$ and observed that velocity of all models is increasing. Also, momentum boundary layer increases for greater values of time. Most importantly the influence of fractional parameter alpha on all the fractional models is drawn in figures 5 and 6. For different values of alpha it is found that hybrid fractional model is explained well the history of the present problem rather than power law (C), exponential law (CF) and Mittag-Leffler law respectively. It is also observed from figure when alpha approaches to 1 the momentum boundary layer thickness of the velocity reduced. From all the graphs it can be seen that as $y \rightarrow \infty$ the velocity asymptotically approaches to zero which satisfies the boundary condition.

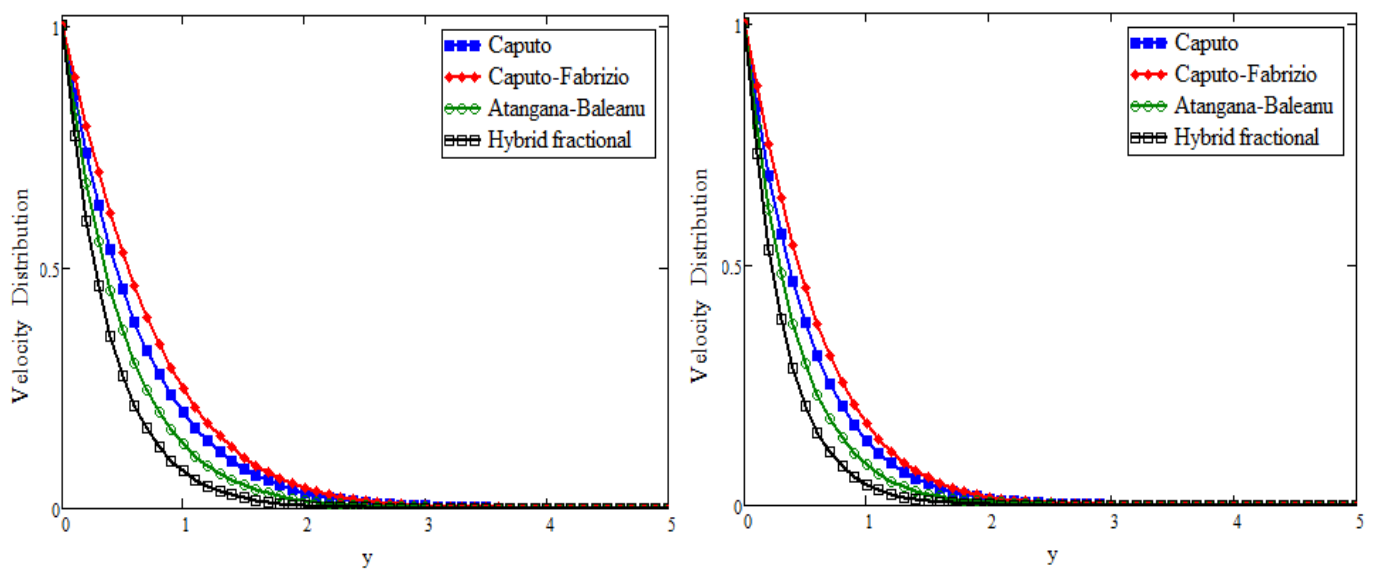


Figure 3.1: Velocities of power law (C), exponential law (CF) and Mittag-Leffler law (ABC) and hybrid fractional derivatives for $t=5$, $Re = 3$ and $Re = 4$ and $\alpha = 0.4$

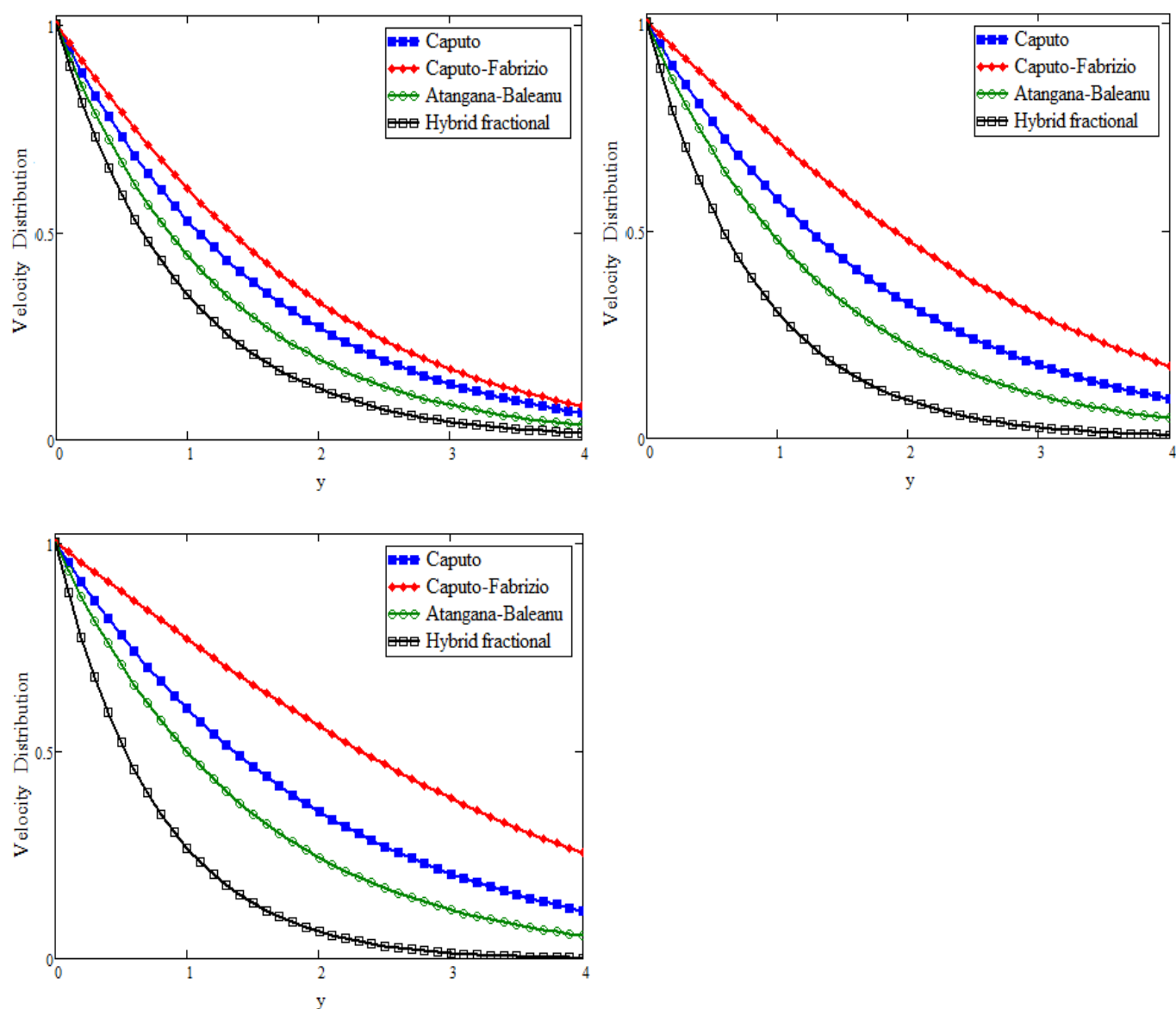


Figure 3.2: Velocities of power law (C), exponential law (CF) and Mittage-Leffler law (ABC) and hybrid fractional derivatives for $t=5, 10$ and 15 , $Re = 1$ and $\alpha = 0.4$

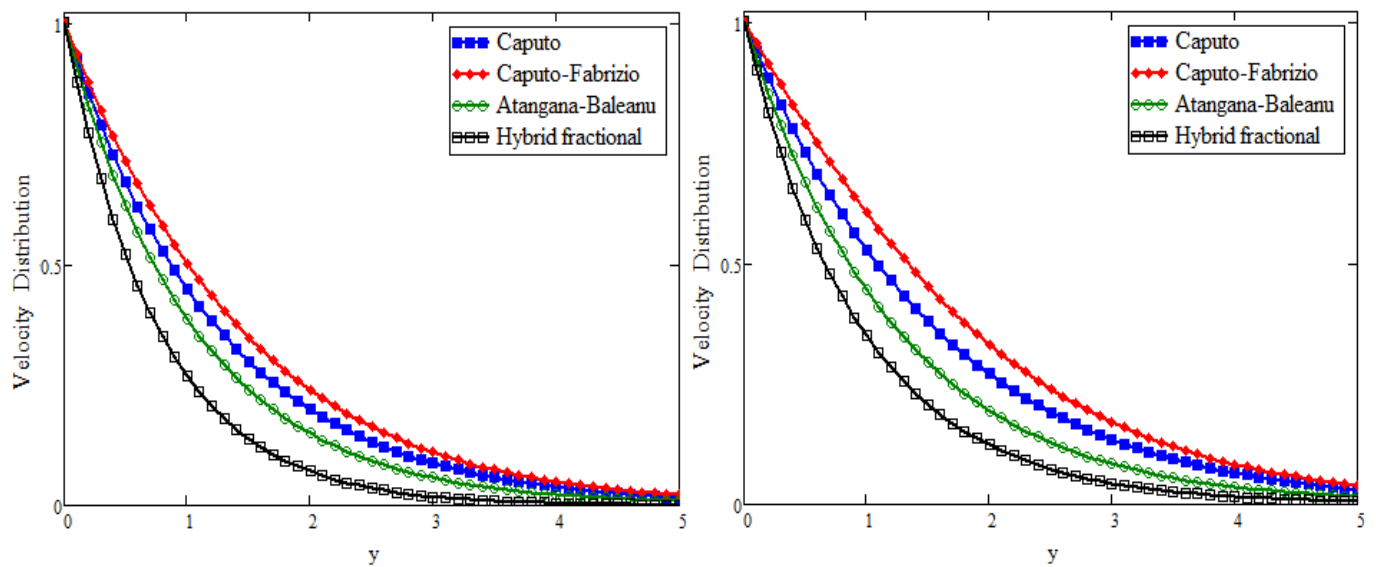


Figure 3.3: Velocities of power law (C), exponential law (CF) and Mittag-Leffler law (ABC) and hybrid fractional derivatives for $\alpha = 0.2$ and $\alpha = 0.4$, $Re = 1$ and $t = 5$

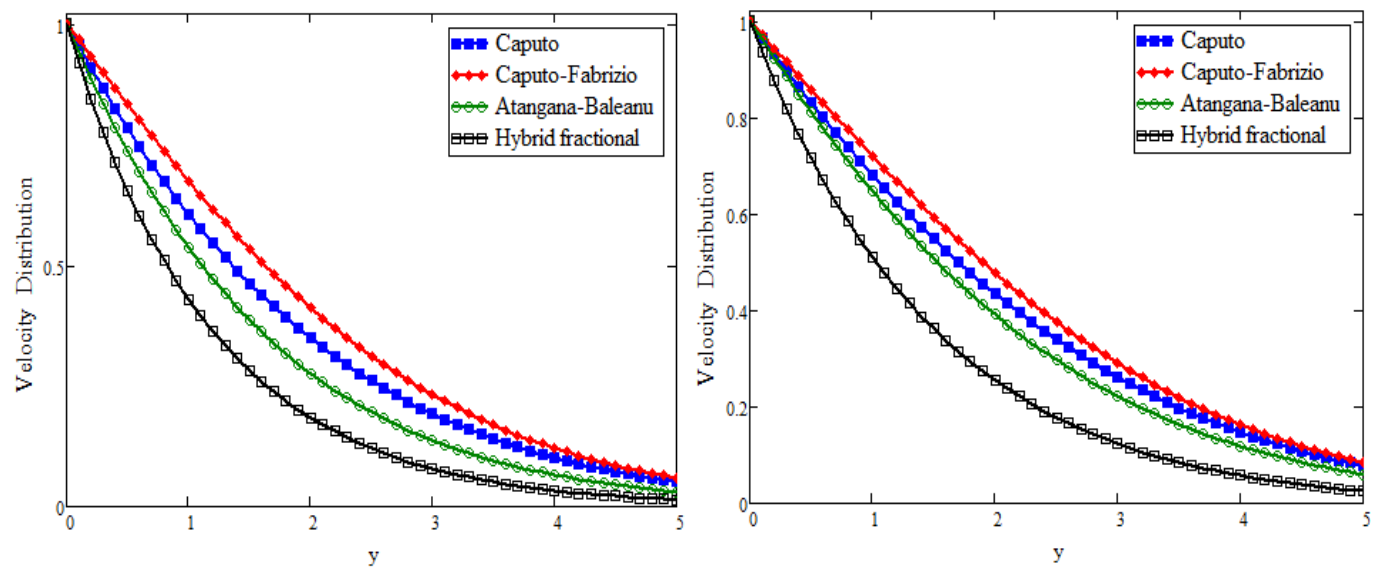


Figure 3.4: Velocities of power law (C), exponential law (CF) and Mittag-Leffler law (ABC) and hybrid fractional derivatives for $\alpha = 0.6$ and $\alpha = 0.8$, $Re = 1$ and $t = 5$

4. Conclusions

This paper deals with the theoretical application of constant proportional Caputo fractional derivative with singular kernel which is a linear combination of conformable fractional derivative and Caputo fractional derivative. One of the well known problem of fluid dynamics; Stokes' first problem is discussed with hybrid fractional derivative by means of Laplace transform method and obtained new analytical solutions which are expressed in terms of series. The main outcomes of the present study are:

- Momentum boundary layer thickness between all the fractional models increases by increasing time while for increasing $\alpha \rightarrow 1$ it reduced.
- In a comparison between constant proportional Caputo and all other existing fractional operators, we concluded that constant proportional Caputo fractional exhibits decay of the velocity function better than them.

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