

# Computing Sharp Bounds for Local Fractional Metric Dimensions of Cycle Related Graphs

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## Abstract

The parameters of approachability, centrality, connectedness and convolution of the aforementioned graphs have been discussed in this note with the help of local fractional metric dimension of wheel related graphs namely, pseudo regular type II, III, generalized, windmill, Dutch windmill, path windmill and wheel windmill graphs. Moreover, the critical analysis of the boundedness and behaviour of the obtained results at infinity has been given at the end of this note in tabulation as well as graphical (3D Plots) form.

**Keywords:** Local Fractional Metric Dimension; Local Resolving Neighbourhoods; Pseudo Regular Graphs and Windmill Graphs

**Mathematics Subject Classification:** 2000 : 05C78

## 1. Introduction

Ever since the advent of distance based dimensions in the strata of Mathematics, many structural properties of graph structures such as connectivity, complexity, vulnerability and mobility has been solved and rectified to great extent. Apart from that different disciplines of sciences particularly robotics and cheminformatics have been benefited by these as well. Suppose  $T = \{t_1, t_2, t_3, \dots, t_p\}$  and  $T \subseteq V(\mathbb{G})$  then  $T$  is forms a set of vertices on which some ordering have been imposed on them. In the same manner, in  $p$ -tuple metric form  $T$  is represented as  $r(f|T) = (d(f, t_1), d(f, t_2), d(f, t_3), \dots, d(f, t_k))$ . The set  $T$  changes into resolving set bearing  $p$  elements for  $\mathbb{G}$  of order  $n$  then the set  $\{r(f|T) | f \in V(\mathbb{G})\}$  is the set having  $n$  distinct  $p$ -tuples. The resolving set with the least cardinality forms the metric basis of  $\mathbb{G}$ .

The resolving sets are the brain child of Slater [14, 15], Harary and Melter [16], as they separately defined this notion and they called it by the metric dimension of  $\mathbb{G}$ . Since then many researchers have gone into the in-depth study of various families of graphs for the evaluation of their metric dimensions. The results for the metric dimensions of path, cycle, peterson and generalized peterson graphs have been found by [17, 18, 19] and [7]. Chartrand *et al* [7], employed metric dimension to solve the integer programming problem (IPP). Afterwards, Currie and Oellermann introduced the notion of fractional metric dimension (FMD) and solved the IPP bearing high accuracy. Arumugam and Mathew [3] after going through many

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new properties of FMD formally defined it. Subsequently, many researchers have gambled their fate in this nexus by considering different graph structures. Particularly, the FMD of graph structures as obtained from cartesian, hierarchial, corona, lexicographic and comb product of connected graph structures can be found in, [4, 9, 10] and [12]. Recently, Liu *et al* [5] calculated the fractional metric dimension of the generalized Jahangir graph  $J_{5,k}$ .

Aisyah *et al.* (2019) gave the notion of local fractional metric dimension (LFMD) and found it for the corona product of two graphs [24]. Recently, Liu *et al.* (2020) computed the local fractional metric dimension of rotationally symmetric and planner graphs. In this note, we are going to evaluate the sharp bounds for the LFMDs of cycle related graphs such as cycle, pseudo regular graphs(type-I and type-II), generalized Dutch windmill graph, generalized path windmill graph and generalized wheel windmill graph. This note continues in the following sequence, Section 1 is for introduction, Section 2 for preliminaries, section 3 concerns with the applications of LFMD, section 4 deals with the resolving neighbourhoods of the aforesaid networks. In section 5, we have calculated the FMD of the graphs under consideration and its criterion is developed as well. In section 6, the boundedness of the LFMD of the graphs are analysed in tabular as well as 3D plot form. Whereas, section 7 is of conclusion.

## 2. Preliminaries

For  $d \in V(\mathbb{G})$  and  $\{e, f\} \subseteq V(\mathbb{G})$  then  $d$  is said to resolve  $\{e, f\}$  if  $d(e, d) \neq d(f, d)$ . The collection of all the vertices like  $d$  forms a set called by resolving neighbourhood. The resolving neighbourhood (RN) of  $\{e, f\}$  is mathematically given by as  $R\{e, f\} = \{d \in V(\mathbb{G}) | d(e, d) \neq d(f, d)\}$ .

Let  $\mathbb{G}(V(\mathbb{G}), E(\mathbb{G}))$  a connected graph with having order  $p$ . A function  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  is called resolving function (RF) of  $\mathbb{G}$  if  $\psi(R\{e, f\}) \geq 1$  if  $\forall e, f \in V(\mathbb{G})$ , where  $\psi(R\{x, y\}) = \sum_{d \in R\{e, f\}} \rho(d)$ . An

RF  $\eta$  of  $\mathbb{G}$  is known as minimal resolving function (MRF) if any function  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that  $\psi \leq \rho$  and  $\rho(d) \neq \psi(d)$  for at least one  $d \in V(\mathbb{G})$ , that is not an RF of  $\mathbb{G}$ . Then LFMD of the graph  $\mathbb{G}$  is given by  $dim_{lfac}(\mathbb{G}) = \min\{|\psi| : \psi \text{ is the MRF of } \mathbb{G}\}$ , where  $|\eta| = \sum_{z \in V(\mathbb{G})} \eta(z)$  [3].

### 2.1. Algorithms for the Graphs Under Consideration

In this part, we present the algorithms that will help to understand the construction of the graphs under consideration.

### 2.2. Pseudo Regular Graphs

A graph  $\mathbb{G}$  which is simple and connected with order  $p$  and number of and size  $q$  is said to be pseudo regular graph if all of its vertices have same average degree. The Average degree can be find by formula as follows.

$$\dot{m}(v_p) = \frac{t_p}{d_p} \quad \forall v \in V(\mathbb{G})$$

### Example :

By using average degree formula, we have found that the average degree number of all of its vertices are same, that is  $\dot{m}(v_p) = 2$ . Here average of all vertices is 2, So given graph is pseudo regular graph.

### 2.3. Type-I

**Step-1** Take any integer non-zero and non-positive number  $s \geq 3$  and first construct a Wheel graph where number of spokes is  $m$ , where  $m$  depends on  $s$  such that  $m = s^2 - 3s + 3$ . Let  $v_0$  is the central vertex of the wheel  $W_{m+1}$ . We may call We have  $d(v_0) = s^2 - 3s + 3$ .

**Step-2**  $\{v_1, v_2, v_3, \dots, v_m\}$  are the vertices of order  $m$  in cycle  $C_m$ . Now next we connect  $s - 3$  new pendent

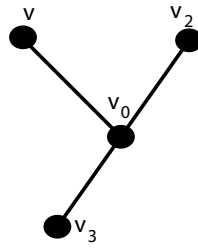


Figure 2.1

vertices with all the other vertices of  $\mathbb{G}$  excluding  $v_0$ . In this case the required graph is pseudo regular graph Type-I. It is denoted by  $\hat{G}_I$ .

**Step-3** Now next We show the cardinality of Vertices and edges as follows.

$$\begin{aligned} |V(\hat{G}_I)| &= (s-3)(s^2-3s+3) + s^3 - 3s + 3 + 1 \\ &= s^3 - 5s^2 + 9s - 5 \end{aligned}$$

For  $s=4$  and  $s=5$  we have the graph ,as shown in figure.

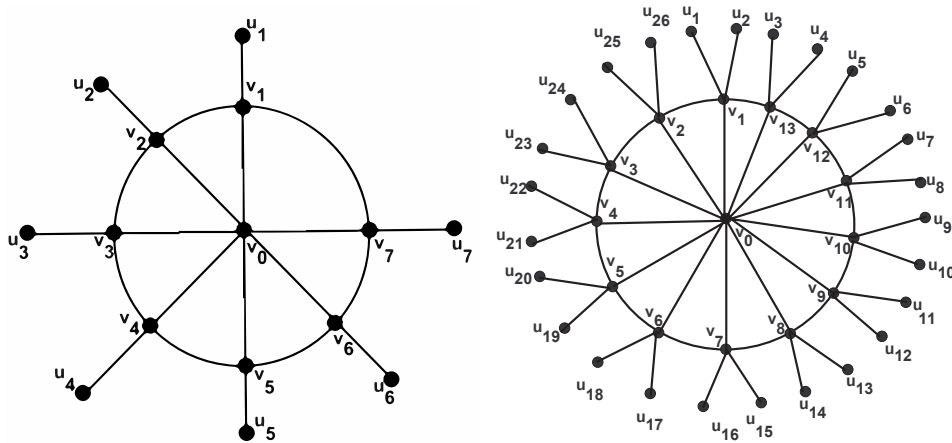


Figure 2.2

## 2.4. Type-II

**Step-1** Take  $s \geq 5$  and then construct a wheel graph where number of spokes is  $m$  and depends on  $s$  such that  $m = s^2 - 3s + 1$ . Let  $v_0$  is the vertex which place in the centre of the wheel graph  $W_{m+1}$ . We may call it as central vertex and we have  $d(v_0) = s^2 - 3s + 1$ .

**Step-2**  $\{v_1, v_2, v_3, \dots, v_m\}$  are the collection of vertices of order  $m$  in cycle  $C_m$ . Now, next we create a new vertex corresponding to each edge which is in the cycle and in a wheel graph. Next new introduced vertices should be connect to the end vertices of each edge in a cycle. Now next we connect  $s-5$  new vertices which is pendent with all the other vertices of  $\mathbb{G}$  excluding  $v_0$  in the cycle  $C_m$  and Wheel Graph. In this case the required graph is pseudo regular graph Type-II. Symbolically, it can be represented by  $\hat{G}_{II}$ .

**Step-3** Now, the cardinality of vertices is as follows.

$$|V(\mathbb{G}_{II})| = s^3 - 6s^2 + 10s - 2$$

For  $s=5$  and  $s=6$  we have the graph ,as shown in figure.

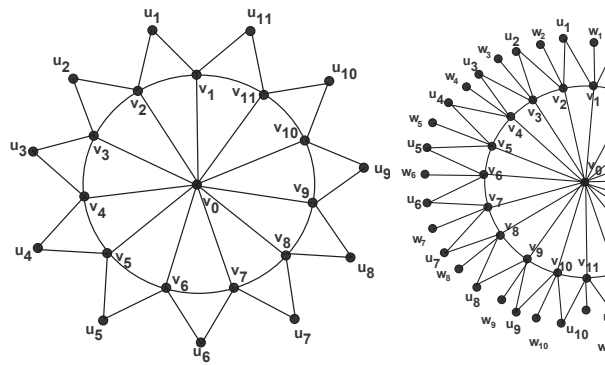


Figure 2.3: Pseudo Regular graph  $G_{II}$   $s=5,6$

## 2.5. Generalized Dutch Windmill Graph

Following is the algorithm for the construction of the generalized Dutch windmill graph.

- Take  $m$  number of cycles each with different lengths  $q_1, q_2, \dots, q_m$  such that  $q_1 \leq q_2 \leq \dots \leq q_m$ .
- Take an isolated vertex  $v$ .
- For  $1 \leq p \leq m$  join/merge the  $q_p$ -th vertex of each cycle with  $v$ .
- The graph so obtained is a generalized Dutch windmill graph.

It is denoted by  $GDW(m; q_1, q_2, \dots, q_m)$ . The vertex and edge sets are given by

$$V(GDW(m; q_1, q_2, \dots, q_m)) = \bigcup_{u=1}^m \{x_1^u, x_2^u, \dots, x_{q_u-1}^u\} \cup \{v\},$$

$$E(GDW(m; q_1, q_2, \dots, q_m)) = \bigcup_{u=1}^m \{x_j^u x_{j+1}^u | 1 \leq j \leq q_u - 1\} \cup \bigcup_{u=1}^m \{vx_1^u, vx_2^u, \dots, vx_{q_u-1}^u\},$$

Therefore from the sets above, we have found that  $|V(GDW(m; q_1, q_2, \dots, q_m))| = \sum_{u=1}^m q_u - m + 1$  and

$$|E(GDW(m; q_1, q_2, \dots, q_m))| = \sum_{u=1}^m q_u.$$

The figure 2.4 shows  $GPW(m; q_1, q_2, \dots, q_m)$ .

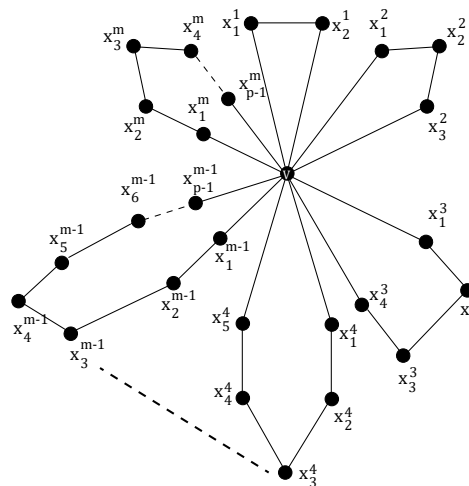


Figure 2.4

$$E(\text{GWW}(m; q_1, q_2, \dots, q_m)) = \bigcup_{u=1}^m \{x_j^u x_{j+1}^u | 1 \leq j \leq u-1\} \cup \bigcup_{u=1}^m \{vx_1^u, vx_2^u, \dots, vx_u^u\}, \text{ Therefore from the}$$

sets above, we have found that  $|V(\mathbb{GWW}(m; q_1, q_2, \dots, q_m))| = \sum_{u=1}^m q_u + 1$  and  $|E(\mathbb{GWW}(m; q_1, q_2, \dots, q_m))| = 2 \sum_{u=1}^m q_u$ . The figure below shows  $\mathbb{GWW}(m; q_1, q_2, \dots, q_m)$ .

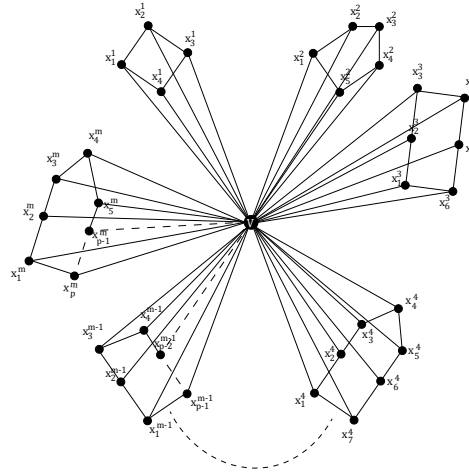


Figure 2.6

### 3. Applications

#### 3.1. Engineering

The advancements in system networking makes the presence of Metric dimension vulnerable. The metric dimension portrayed by Slatter [14] and Harary and Melter [16] provides software engineers with opportunities encounter the problem in which an interpolator has to decide for adopting an occupation in particular region.

The aids like these helps the engineers in installation of robots in restaurants, factories and in different public service units [21]. These tools also helps in rectifying the problems such as example handling, picture handling and handling progressive information structures [20], thereby helps the engineers in designing a smart systems.

#### 3.2. Chemistry

The diagram of a chemical compound as presented by [8] helps the pharmacists and chemists in the formation of different drugs and finding antibiotic that can not only treat bacteria but can also destroy the viruses having compatible structures. This helps the pharmacist in an error free drug formation. For having an insight into the problems like these we refer to [14, 20, 22] and [23].

### 4. Local Resolving Neighbourhoods

In this section of the paper, we will present the local resolving neighbourhoods of the graphs under consideration.

#### 4.1. Local Resolving Neighbourhoods of $C_m$

Here we will present the local resolving neighbourhoods of  $C_m$ .

**Lemma 4.1.1:** For  $m \geq 4$ ,  $m \equiv 0 \pmod{2}$  and  $V(\mathbb{G}) \cong \mathbb{C}_m$ . Then  $|R\{x_j x_{j+1}\}| = |V(\mathbb{G})| = m$  and

$$|\bigcup_{j=1}^m R_j| = |V(\mathbb{G})| = m.$$

**Proof:** Since  $\mathbb{C}_m$  is a bipartite graph for the case under consideration, so all of its local resolving

neighbourhoods will be equal to its vertex set. Therefore,  $R\{x_j x_{j+1}\} = V(\mathbb{G})$ ,  $\bigcup_{j=1}^m R_j = V(\mathbb{G})$  and  $|R_j| = |\bigcup_{j=1}^m R_j| = m$ .

**Lemma 4.1.2:** For  $m \geq 3$ ,  $m \equiv 1 \pmod{2}$  and  $V(\mathbb{G}) \cong \mathbb{C}_m$ . Then  $|R\{x_j x_{j+1}\}| = m - 1$ ,  $|\bigcup_{j=1}^m R_j| = |V(\mathbb{G})| = m$ .

**Proof:** The local resolving neighbourhoods for the case under the discussion is given by:

$R\{x_j x_{j+1}\} = V(\mathbb{G}) - \{x_k | k \equiv j + \frac{m+1}{2} \pmod{m}\}$  Therefore,  $|R\{x_j x_{j+1}\}| = m - 1$ ,  $|\bigcup_{j=1}^m R_j| = |V(\mathbb{G})|$ .

#### 4.2. Local Resolving Neighbourhoods of $\mathbb{G}_I$ and $\mathbb{G}_{II}$

In this section, we present some the results regarding the resolving neighbourhoods of  $\mathbb{G}_{II}$  and  $\mathbb{G}_{III}$ .

**Lemma 4.2.1:** Suppose  $\mathbb{G} \cong \mathbb{G}_{II}$  taking any non-negative number  $s \geq 4$  and  $n = s^2 - 3s + 3$ .

(a)  $|R_u| = |R\{v_m, v_{m+1}\}| = 4s - 7$  and  $|\bigcup_{u=1}^n R_u| = s^3 - 5s^2 + 9s - 6$ ,

(b)  $|R_t| < |R\{v_0, v_l\}|$  and  $|R\{v_0, v_l\} \cap \bigcup_{l=1}^n R_l| \geq |R_l|$  with  $v_{m+1} = v_1$ .

(c)  $|R_l| < |R\{v_l, w_k^{v_m}\} \cap \bigcup_{l=1}^m R_l| \geq |R_l|$

**Proof:** (a) The local resolving neighbourhood for  $v_m v_{m+1}$  are:

$R\{v_m, v_{m+1}\} = V(\mathbb{G}) - \{v_0\} \cup \{v_u | u = m + 3, m + 4, \dots, m + n - 4 \pmod{m}\} \cup \{w_k v_u | 1 \leq s \leq s - 3\}$  we see that  $\bigcup_{u=1}^n R_u = V(\mathbb{G}) = V(\mathbb{G}_{II}) - \{v_0\}$  and  $|\bigcup_{u=1}^n R_u| = q^3 - 5q^2 + 9q - 6$

(b) The Local resolving neighbourhood for  $v_0 v_m$  is as follows, Now  $1 \leq l \leq n$  and  $1 < k < s - 3$   $R\{v_0, v_m\} = V(\mathbb{G}) - \{v_u | u = m - 1, m + 2, \pmod{m}\} \cup \{w_k v_u | 1 < k < s - 3\}$  and  $|R\{v_0, v_m\} \cap \bigcup_{u=1}^n R_u| \geq |R_u|$ .

(c) The local resolving neighbourhood of  $R\{v_l, w_k^{v_m}\} = V(\mathbb{G})$  is :

$R\{v_l, w_k^{v_m}\} = V(\mathbb{G})$ ,

where  $1 \leq k \leq s - 3$ . Thus it is evident from the above that  $|R_u| < |R\{v_l, w_k^{v_m}\}|$  and  $|R\{v_l, w_k^{v_m}\} \cap \bigcup_{u=1}^n R_u| \geq |R_u|$ .

**Lemma 4.2.2:** Suppose  $\mathbb{G} \cong \mathbb{G}_{III}$  considering any non-negative number  $s \geq 4$  and  $n = s^2 - 3s + 1$ .

(a)  $|R_u| = |R\{v_m, v_{m+1}\}| = 4s + 7$  and  $|\bigcup_{l=1}^n R_u| = s^3 - 6s^2 + 10s - 3$

(b)  $|R_u| < |R\{v_0, v_m\}| = |R\{v_m u_m\}| = |R\{v_m u_{m-1}\}|$  and  $|R\{v_0, v_m\} \cap \bigcup_{u=1}^n R_u| = |R\{v_m w_m\} \cap \bigcup_{u=1}^n R_u| = |R\{v_m w_{m-1}\} \cap \bigcup_{u=1}^n R_u| \geq |R_u|$  with  $v_{m+1} = v_1$  and  $u_0 = u_m$ .

(c)  $|R_u| < |R\{v_m, x_k^{v_m}\}|$  and  $|R\{v_m, u_j\} \bigcup_{l=1}^m R_u| \geq |R_u|$ , where  $1 \leq k \leq s - 5$ .

**Proof:**

(a) The local resolving neighbourhood for  $v_m v_{m+1}$  as follows.

$R(v_m, v_{m+1}) = V(\mathbb{G}) - \{v_0\} \cup \{v_m, v_u | u = m+3, m+4, \dots, m+n-3 \pmod n\} \cup \{x_k^{v_m} | 1 \leq k \leq s-5\}$ . We have found that  $\bigcup_{l=1}^m R_u = V(\mathbb{G}) = V(\mathbb{G}) - \{v_0\}$  and  $|\bigcup_{l=1}^m R_u| = s^3 - 6s^2 + 10q - 3$ .

(b) The Local resolving neighbourhood for  $\{v_0 v_m\}, \{v_m, w_m\}$  and  $\{v_m, w_{m-1}\}$  are as follows,

$R\{v_0, v_m\} = V(\mathbb{G}) - \{v_u | u \equiv m-1, m+1 \pmod m\} \cup \{w_u | u \equiv m+1, m-2 \pmod m\} \cup \{x_k^{v_m} | 1 \leq k \leq s-5\}$ ,  
 $R\{v_m, w_m\} = V(\mathbb{G}) - \{v_u | u \equiv m+1, m+2 \pmod m\} \cup \{w_u | u \equiv m+1, m+2 \pmod m\} \cup \{x_k^{v_m} | 1 \leq k \leq s-5\}$ , and

$R\{v_m, w_{m-1}\} = V(\mathbb{G}) - \{v_u | u \equiv m-1, m-2 \pmod m\} \cup \{w_u | u \equiv m-1, m-2 \pmod m\} \cup \{x_k^{v_m} | 1 \leq k \leq s-5\}$  respectively.

It is observed that  $|R\{v_0 v_m\}| = |R\{v_m, w_m\}| = |R\{v_m, w_{m-1}\}| = s^3 - 6s^2 + 8s + 4 > |R_u|$ . Thus  $|R\{v_0, v_m\} \cap \bigcup_{u=1}^n R_u| = |R\{v_m w_m\} \cap \bigcup_{u=1}^n R_u|$   
 $= |R\{v_m w_{m-1}\} \cap \bigcup_{u=1}^n R_u| \geq |R_u|$ .

(c) The local resolving neighbourhood for  $\{v_m, x_k^{v_m}\}$  with  $1 \leq k \leq s-5$  is

$R\{v_m, x_k^{v_m}\} = V(\mathbb{G})$ . As we know that  $V(\mathbb{G}) = s^3 - 6s^2 + 10s - 2 > |R_u|$ . Therefore,  $|R\{v_m, x_k^{v_m}\} \cap \bigcup_{u=1}^n R_u|$ .

#### 4.3. Local Resolving Neighbourhoods of $\mathbb{GDW}(m; q_1, q_2, \dots, q_p)$ , $\mathbb{GPW}(m; q_1, q_2, \dots, q_p)$ and $\mathbb{GWW}(m; q_1, q_2, \dots, q_p)$

In this section of the note, we are presenting the local resolving neighbourhoods of  $\mathbb{GDW}(m; q_1, q_2, \dots, q_p)$ ,  $\mathbb{GPW}(m; q_1, q_2, \dots, q_p)$  and  $\mathbb{GWW}(m; q_1, q_2, \dots, q_p)$ .

**Lemma 4.3.1(a)** Suppose  $\mathbb{G} \cong \mathbb{GDW}(m; q_1, q_2, \dots, q_m)$  for any  $m$ ,  $1 \leq i \leq m$ ,  $q_i \geq 3$  and  $q \equiv 0 \pmod 2$ . Then  $R\{uv\} = V(\mathbb{G})$ ,  $\forall uv \in E(\mathbb{G})$ ,

(b) For any  $m$ ,  $1 \leq i \leq m$ ,  $q_i \geq 3$  and  $q \equiv 1 \pmod 2$ . Then:

(i) For  $1 \leq j \leq m$ ,  $|R_t| = |R\{x_r^j x_s^j\}| = \alpha - 1$  and  $|\bigcup_{t=1}^p R_t| = \sum_{i=1}^m q_i - m$ . where  $r = \frac{q-1}{2}$

$\alpha = \min\{q_i | 1 \leq i \leq m\}$  and  $s = \frac{q+1}{2}$ ,

(ii) For  $1 \leq j \leq m$ ,  $|R_t| < |R\{x_{r_i}^j x_{s_i}^j\}| = \beta - 1$ ,  $|\bigcup_{t=1}^p R_t| = \sum_{i=1}^m q_i - m$ . where  $r_i = \frac{q_i-1}{2}$

$\beta = \{q_i | 1 \leq i \leq m\} - \{\alpha\}$  and  $s_i = \frac{q_i+1}{2}$ ,

(iii) For  $1 \leq j \leq m$ ,  $1 \leq i \leq r_i - 1$  and  $s_i \leq k \leq q_i - 2$ ,

$|R_t| < |R\{x_i^j x_{i+1}^j\}| = |R\{x_k^j x_{k+1}^j\}| = |R\{v x_1^j\}| = |R\{v x_{p-1}^j\}| = p(q-1)$  and  $|R\{x_i^j x_{i+1}^j\} \bigcup_{t=1}^p R_t| = |R\{x_k^j x_{k+1}^j\} \bigcup_{t=1}^p R_t| =$

$|R\{v x_1^j\} \bigcup_{t=1}^p R_t| = |R\{v x_{p-1}^j\} \bigcup_{t=1}^p R_t| \geq |R_t|$ .

**Proof:**

(a) The given statement is true as the graph will be a bipartite.

(b)

(i) The local resolving neighbourhood of  $x_r^j x_s^j$  is

$R\{x_r^j x_s^j\} = V(\mathbb{G}) - \{v\} \cup \{x_k^l, x_k^m | 1 \leq l \leq j-1, j+1 \leq n \leq m \wedge 1 \leq k \leq \alpha-1\}$ . As we can see that

$|R\{x_r^j x_s^j\}| = \alpha - 1$ , thus  $|\bigcup_{t=1}^p R_t| = \sum_{i=1}^m q_i - m$ .

(ii) The local resolving neighbourhood of  $x_{r_i}^j x_{s_i}^j$  is

$R\{x_{r_i}^j x_{s_i}^j\} = V(\mathbb{G}) - \{v\} \cup \{x_k^l, x_k^m | 1 \leq l \leq j-1, j+1 \leq n \leq m \wedge 1 \leq k \leq q_i-1\}$ . As we can see that

$$|R\{x_r^j x_s^j\}| = \beta - 1, \text{ thus } \left| \bigcup_{t=1}^p R_t \right| = \sum_{i=1}^m q_i - m.$$

(iii) The local resolving neighbourhoods of  $x_i^j x_{i+1}^j$ ,  $x_k^j x_{k+1}^j$ ,  $vx_1^j$  and  $vx_{q_i-1}^j$  are

$$R\{x_i^j x_{i+1}^j\} = V(\mathbb{G}) - \{x_l^j | l \equiv i + s_i(\text{mod } q_i)\},$$

$$R\{x_k^j x_{k+1}^j\} = V(\mathbb{G}) - \{x_l^j | l \equiv k - s_i(\text{mod } q_i)\},$$

$$R\{vx_1^j\} = V(\mathbb{G}) - \{x_s^j\},$$

and  $R\{vx_{p-1}^j\} = V(\mathbb{G}) - \{x_r^j\}$  respectively. It is clear from the above that  $|R\{x_i^j x_{i+1}^j\}| = |R\{x_k^j x_{k+1}^j\}| =$

$$|R\{vx_1^j\}| = |R\{vx_{q_i-1}^j\}| = p(q-1) > |R_t| \text{ and } |R\{x_i^j x_{i+1}^j\} \bigcup_{t=1}^p R_t| = |R\{x_k^j x_{k+1}^j\} \bigcup_{t=1}^p R_t| = |R\{vx_1^j\} \bigcup_{t=1}^p R_t| =$$

$$|R\{vx_{p-1}^j\} \bigcup_{t=1}^p R_t| \geq |R_t|.$$

**Lemma 4.3.2** (a) Suppose  $\mathbb{G} \cong \mathbb{GPW}(m; q_1, q_2, \dots, q_m)$ , for any  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $q_p \geq 6$ .  $|R_u| = |R\{x_1^p x_2^p\}| = |R\{x_{n-1}^p x_2^p\}| = 3$  and  $\bigcup_{u=1}^p R_u = \bigcup_{u=1}^m \{x_1^u, x_2^u, x_3^u, x_{q_u-2}^u, x_{q_u-1}^u, x_{q_u-2}^u\}$  and  $\bigcup_{u=1}^p R_u = 6m$ ,

(b) For  $2 \leq j \leq q_p - 2$ ,  $|R_u| < |R\{x_j^p x_{j+1}^p\}|$  and  $|R\{x_j^p x_{j+1}^p\} \cap \bigcup_{u=1}^p R_u| \geq |R_u|$ , (c)  $|R_u| < |R\{vx_1^p\}| = |R\{vx_n^p\}|$

and  $|R\{vx_1^p\} \cap \bigcup_{u=1}^p R_u| = |R\{vx_n^p\} \cap \bigcup_{u=1}^p R_u| \geq |R_u|$ ,

(d) For  $2 \leq j \leq q_p - 1$ ,  $|R_u| < |R\{vx_j^p\}|$  and  $|R\{vx_n^p\} \cap \bigcup_{u=1}^p R_u| \geq |R_u|$ .

**Proof:**

(a) The local resolving neighbourhoods of  $x_1^p x_2^p$  and  $x_{q_p-1}^p x_{q_p}^p$  are

$R\{x_1^p x_2^p\} = \{x_1^p, x_2^p, x_3^p\}$  and  $R\{x_{q_p-1}^p x_{q_p}^p\} = \{x_{q_p-2}^p, x_{q_p-1}^p, x_{q_p}^p\}$  respectively. It is clear from the sets that

$$|R\{x_1^p x_2^p\}| = |R\{x_{n-1}^p x_2^p\}| = 3 \text{ and } \bigcup_{u=1}^p R_u = 6m.$$

(b) The local resolving neighbourhood of  $x_j^p x_{j+1}^p$  is

$R\{x_j^p x_{j+1}^p\} = \{x_k^p | k \equiv j-1, j, j+1, j+2 (\text{mod } q_p)\}$ . Clearly,  $|R\{x_j^p x_{j+1}^p\}| = |V(\mathbb{G})| - 4$  and  $|R\{x_j^p x_{j+1}^p\} \cap \bigcup_{u=1}^p R_u| \geq |R_u|$ .

(c) The local resolving neighbourhoods of  $vx_1^p$  and  $vx_n^p$  are

$R\{vx_1^p\} = V(\mathbb{G}) - \{x_2^p\}$  and  $R\{vx_{q_p}^p\} = V(\mathbb{G}) - \{x_{q_1-1}^p\}$  respectively. As we can see that  $|R\{vx_1^p\}| = |R\{vx_{q_p-1}^p\}| = |V(\mathbb{G})| - 1$  and  $|R\{vx_1^p\} \cap \bigcup_{u=1}^p R_u| = |R\{vx_{q_p}^p\} \cap \bigcup_{u=1}^p R_u| \geq |R_u|$ .

(d) The local resolving neighbourhood of  $vx_j^p$  for  $2 \leq j \leq q_p - 1$  is

$R\{vx_j^p\} = V(\mathbb{G}) - \{x_k^p | k \equiv j-1, j+1 (\text{mod } q_p)\}$ . It is clear from the set that  $|R\{vx_j^p\}| = |V(\mathbb{G})| - 2$  and  $|R\{vx_j^p\} \cap \bigcup_{u=1}^p R_u| \geq |R_u|$ .

**Lemma 4.3.3** Suppose  $\mathbb{G} \cong \mathbb{GWW}(m; q_1, q_2, \dots, q_m)$ , for any  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $q_p \geq 6$  and  $1 \leq j \leq q_p$ .

(a) For  $\hat{q}_p = \min\{q_p | 1 \leq p \leq m\}$ ,  $1 \leq j \leq \hat{q}_p$  with  $\hat{q}_p + 1 = 1$  and  $1 \leq \hat{p} \leq p$ ,

$$|R_u| = |R\{x_j^p x_{j+1}^p\}| = \begin{cases} q_p & \text{for } q_p \equiv 0(\text{mod } 2), \\ q_p - 1 & \text{for } q_p \equiv 1(\text{mod } 2) \end{cases}$$

,

$$\bigcup_{u=1}^{\dot{p}} R_u = \begin{cases} \bigcup_{u=1}^{\dot{p}} \{x_k^u | 1 \leq k \leq q_p\} & \text{for } q_p \equiv 0(\text{mod } 2), \\ \bigcup_{u=1}^{\dot{p}} (\{x_k^u | 1 \leq k \leq q_p\} - \{x_r^u\}) & \text{for } q_p \equiv 1(\text{mod } 2) \end{cases}$$

and

$$|\bigcup_{u=1}^{\dot{p}} R_u| = \begin{cases} \dot{p}q_u & \text{for } q_p \equiv 0(\text{mod } 2), \\ \dot{p}(q_u - 1) & \text{for } q_p \equiv 1(\text{mod } 2) \end{cases}$$

. Where  $\dot{p}$  is the number of  $q_p$  in  $\mathbb{G}$  and  $r = \frac{q_p+1}{2}$ .

(b) For  $1 \leq j \leq q_p$  and  $1 \leq n \leq p - \dot{p}$ ,  $|R_u| < |R\{x_j^n x_{j+1}^n\}|$  and  $|R\{x_j^n x_{j+1}^n\} \cap \bigcup_{u=1}^{\dot{p}} R_u| \geq |R_u|$ ,

(c) For  $1 \leq j \leq q_p$ ,  $|R_u| < |R\{vx_j^p\}|$  and  $|R\{vx_j^p\} \cap \bigcup_{u=1}^{\dot{p}} R_u| \geq |R_u|$ .

**Proof:**

(a) The local resolving neighbourhoods of  $x_j^p x_{j+1}^p$  are

$$R_u = R\{x_j^p x_{j+1}^p\} \begin{cases} \{x_k^p | 1 \leq k \leq q_p\} & \text{for } q_p \equiv 0(\text{mod } 2), \\ \{x_k^p | 1 \leq k \leq q_p\} - \{x_r^p\} & \text{for } q_p \equiv 1(\text{mod } 2) \end{cases}$$

It is clear from the sets that

$$|R_u| = |R\{x_j^p x_{j+1}^p\}| = \begin{cases} q_p & \text{for } q_p \equiv 0(\text{mod } 2), \\ q_p - 1 & \text{for } q_p \equiv 1(\text{mod } 2) \end{cases}$$

,

and

$$|\bigcup_{u=1}^{\dot{p}} R_u| = \begin{cases} \dot{p}q_u & \text{for } q_p \equiv 0(\text{mod } 2), \\ \dot{p}(q_u - 1) & \text{for } q_p \equiv 1(\text{mod } 2) \end{cases}$$

.

(b) The local resolving neighbourhood of  $x_j^n x_{j+1}^n$  is

$R\{x_j^n x_{j+1}^n\} = \{x_k^n | k \equiv j-1, j, j+1, j+2 \pmod{q_p}\}$ . Clearly,  $|R\{x_j^n x_{j+1}^n\}| = |V(\mathbb{G})| - 4$  and  $|R\{x_j^n x_{j+1}^n\} \cap \bigcup_{u=1}^{\dot{p}} R_u| \geq |R_u|$ .

(c) The local resolving neighbourhoods of  $vx_j^p$  are

$R\{vx_j^p\} = V(\mathbb{G}) - \{x_k^p | k \equiv j-1, j+1 \pmod{q_p}\}$ . It is clear from the set that  $|R\{vx_j^p\}| = |V(\mathbb{G})| - 2$  and  $|R\{vx_j^p\} \cap \bigcup_{u=1}^{\dot{p}} R_u| \geq |R_u|$ .

## 5. Local Fractional Metric Dimension

In this patch of the paper, we will calculate the *LFMD* of the graphs under consideration and present some supportive results towards their calculation.

**Lemma 5.1.1** Suppose  $\mathbb{G} \cong \mathbb{G}_I$  and  $R = \{R_u, \acute{R}_l | u \in T \wedge l \in L\}$  be the collection of their local resolving neighbourhoods such that  $|R_u| = \omega \leq |\acute{R}_k|$  and  $|R_u \cap \bigcup_{k=1}^L R_k| \geq \omega$ . For any  $R_j, R_l \in R_u$  and  $R_j \cap R_l \neq \phi$ . Then

$$\dim_{lfrac}(\mathbb{G}) < \sum_{u=1}^{\delta} \frac{1}{\omega},$$

where  $\delta = |\bigcup_{u \in T} R_u|$  and  $R_u$  is the minimum resolving neighbourhood.

**Proof:** As  $R_j \cap R_l \neq \phi$ , hence, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that

$$\psi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^{\delta} R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^{\delta} R_u \end{cases}$$

As all the minimum resolving neighbourhoods of  $\mathbb{G}$  are not pairwise disjoint, hence  $\dim_{lfrac}(\mathbb{G})$  will be strictly less then the sum of  $\frac{1}{\omega}$ . Therefore,

$$\dim_{lfrac}(\mathbb{G}) < \sum_{u=1}^{\delta} \frac{1}{\omega},$$

**Lemma 5.1.2** Suppose  $\mathbb{G} \cong \mathbb{G}_{II}$  and  $R = \{R_u, \acute{R}_l | u \in T \wedge l \in L\}$  be the collection of their local resolving neighbourhoods such that  $|R_u| = \omega \leq |\acute{R}_k|$  and  $|R_u \cap \bigcup_{k=1}^L R_k| \geq \omega$ . For any  $R_j, R_l \in R_u$  and  $R_j \cap R_l \neq \phi$ . Then

$$\dim_{lfrac}(\mathbb{G}) < \sum_{u=1}^{\delta} \frac{1}{\omega},$$

where  $\delta = |\bigcup_{u \in T} R_u|$  and  $R_u$  is the minimum resolving neighbourhood.

**Proof:** As  $R_j \cap R_l \neq \phi$ , hence, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that

$$\psi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^{\delta} R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^{\delta} R_u \end{cases}$$

As all the minimum resolving neighbourhoods of  $\mathbb{G}$  are not pairwise disjoint, hence  $\dim_{lfrac}(\mathbb{G})$  will be strictly less then the sum of  $\frac{1}{\omega}$ . Therefore,

$$\dim_{lfrac}(\mathbb{G}) < \sum_{u=1}^{\delta} \frac{1}{\omega},$$

**Lemma 5.1.3** Suppose  $\mathbb{G} \cong GDW(m; q_1, q_2, \dots, q_m)$  and  $R = \{R_u, \dot{R}_l | u \in T \wedge l \in L\}$  be the collection of their local resolving neighbourhoods such that  $|R_u| = \omega \leq |\dot{R}_k|$  and  $|R_u \cap \bigcup_{k=1}^L R_k| \geq \omega$ . For any  $R_j, R_l \in R_u$  and  $R_j \cap R_l = \phi$ . Then

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega},$$

where  $\delta = |\bigcup_{u \in T} R_u|$  and  $R_u$  is the minimum resolving neighbourhood.

**Proof:** As  $R_j \cap R_l = \phi$ , hence, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that

$$\psi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^{\delta} R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^{\delta} R_u \end{cases}$$

As all the minimum resolving neighbourhoods of  $\mathbb{G}$  are pairwise disjoint, thus,  $\dim_{lfrac}(\mathbb{G})$  and sum of the weights  $\frac{1}{\omega}$  will be equal to both. Hence,

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega},$$

**Lemma 5.1.3** Suppose  $\mathbb{G} \cong GPW(m; q_1, q_2, \dots, q_m)$  and  $R = \{R_u, \dot{R}_l | u \in T \wedge l \in L\}$  be the collection of their local resolving neighbourhoods such that  $|R_u| = \omega \leq |\dot{R}_k|$  and  $|R_u \cap \bigcup_{k=1}^L R_k| \geq \omega$ . For any  $R_j, R_l \in R_u$  and  $R_j \cap R_l = \phi$ . Then

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega},$$

where  $\delta = |\bigcup_{u \in T} R_u|$  and  $R_u$  is the minimum resolving neighbourhood.

**Proof:** Since  $R_j \cap R_l = \phi$ , therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that

$$\psi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^{\delta} R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^{\delta} R_u \end{cases}$$

Since all the minimum resolving neighbourhoods of  $\mathbb{G}$  are not pairwise overlapping, therefore,  $\dim_{lfrac}(\mathbb{G})$  is equal to the sum of the weights  $\frac{1}{\omega}$ . Hence,

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega},$$

**Lemma 5.1.4** Suppose  $\mathbb{G} \cong GWW(m; q_1, q_2, \dots, q_m)$  and  $R = \{R_u, \dot{R}_l | u \in T \wedge l \in L\}$  be the collection of their local resolving neighbourhoods such that  $|R_u| = \omega \leq |\dot{R}_k|$  and  $|R_u \cap \bigcup_{k=1}^L R_k| \geq \omega$ . For any  $R_j, R_l \in R_u$  and  $R_j \cap R_l = \phi$ . Then

$$dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega},$$

where  $\delta = |\bigcup_{u \in T} R_u|$  and  $R_u$  is the minimum resolving neighbourhood.

**Proof:** Since  $R_j \cap R_l = \emptyset$ , therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that

$$\psi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^{\delta} R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^{\delta} R_u \end{cases}$$

Since all the minimum resolving neighbourhoods of  $\mathbb{G}$  are not pairwise overlapping, therefore,  $dim_{lfrac}(\mathbb{G})$  is equal to the sum of the weights  $\frac{1}{\omega}$ . Hence,

$$dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega},$$

**Theorem 5.1.1** If  $\mathbb{G} \cong C_m$  then

$$dim_{lfrac}(\mathbb{G}) = \frac{m}{m-1}.$$

**Proof: Case 1:** When  $n = 3$

We have the following local resolving neighbourhoods of  $C_3$

$$R_1 = R_{x_1x_2} = V(\mathbb{G}) - x_3, R_2 = R_{x_2x_3} = V(\mathbb{G}) - x_1,$$

$$R_3 = R_{x_1x_3} = V(\mathbb{G}) - x_2.$$

We can see from the above local resolving neighbourhoods that each of them is having a cardinality of 2.

Thus,  $\bigcup_{u=1}^3 R_u = V(\mathbb{G})$  and  $|\bigcup_{u=1}^3 R_u| = 3$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning all the vertices of  $\mathbb{G}$  by  $\frac{1}{2}$  and summing up all we get:

$$dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^m \frac{1}{2} = \frac{3}{2}.$$

**Case 2:**

(i) When  $n = 5$

We have the following local resolving neighbourhoods of  $C_3$

$$R_1 = R_{x_1x_2} = V(\mathbb{G}) - x_3, R_2 = R_{x_2x_3} = V(\mathbb{G}) - x_4,$$

$$R_3 = R_{x_3x_4} = V(\mathbb{G}) - x_5, R_4 = R_{x_4x_5} = V(\mathbb{G}) - x_1,$$

$$R_5 = R_{x_1x_5} = V(\mathbb{G}) - x_2. \text{ It can be observed from the above local resolving neighbourhoods that each}$$

of them is having a cardinality of 4. Thus,  $\bigcup_{u=1}^5 R_u = V(\mathbb{G})$  and  $|\bigcup_{u=1}^5 R_u| = 5$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning all the vertices of  $\mathbb{G}$  by  $\frac{1}{4}$  and summing up all we get:

$$dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^m \frac{1}{4} = \frac{5}{4}.$$

(ii) For  $n \geq 5$ :

The local resolving neighbourhoods for  $C_m$  is  $R_u = R\{x_i x_{i+1}\}$ . As we have seen that  $|R_u| = |R\{x_i x_{i+1}\}| = m - 1$  and  $|\bigcup_{u=1}^m R_u| = m$ . Now, let  $\omega = |R_u|$  and  $\delta = |\bigcup_{u=1}^m R_u|$ . Now we define a mapping  $\psi$  such that

Clearly,  $|R_t| = |R\{x_i x_{i+1}\}| = m - 1$  and  $|\bigcup_{t=1}^m R_t| = m$ . Since each of the local so now we define a mapping  $\psi$  such that.

$$\psi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

Now assigning  $\frac{1}{\omega}$  to all the vertices of  $C_m$ . Since all the  $R_u$  are pairwise overlapping so there exist another  $\dots \psi$  such that  $\dots \psi \leq \psi$ . Thus, we have:

$$\dim_{lfac}(C_m) \leq \sum_{u=1}^m \frac{1}{\omega} \leq \frac{m}{m-1}.$$

Since we are considering the case when  $m \equiv 1 \pmod{2}$  that means the  $LFMD$  of  $C_m$  can never be 1. In that case that will be a bipartite graph. Hence  $\dim_{lfac}(C_m) \geq \sum_{u=1}^m \frac{1}{\omega} \geq \frac{m}{m-1}$ . Therefore, from both the cases we come up to the following conclusion

$$\dim_{lfac} = \frac{m}{m-1}.$$

**Theorem 5.1.2:** If  $\mathbb{G} \cong G_I$ , for  $s \geq 4$  and  $m = s^2 - 3s + 3$ . Then  $\dim_{Lf}(G_I) < \frac{s^3 - 5s^2 + 9s - 6}{4s - 7}$ .

**Proof:**

**Case I**

For  $s = 4$  and  $n = 7$ .

The local resolving neighbourhood are as follows :

| Resolving Neighbourhoods       | Elements                                                   |
|--------------------------------|------------------------------------------------------------|
| $\bar{R}_1 = R_L(v_0, v_1)$    | $V(\mathbb{G}) - \{v_2, v_7\} \cup \{u_2, u_7\}$           |
| $\bar{R}_2 = R_L(v_0, v_2)$    | $V(\mathbb{G}) - \{v_1, v_3\} \cup \{u_1, u_3\}$           |
| $\bar{R}_3 = R_L(v_0, v_3)$    | $V(\mathbb{G}) - \{v_2, v_4\} \cup \{u_2, u_4\}$           |
| $\bar{R}_4 = R_L(v_0, v_4)$    | $V(\mathbb{G}) - \{v_3, v_5\} \cup \{u_3, u_5\}$           |
| $\bar{R}_5 = R_L(v_0, v_5)$    | $V(\mathbb{G}) - \{v_4, v_6\} \cup \{u_4, u_6\}$           |
| $\bar{R}_6 = R_L(v_0, v_6)$    | $V(\mathbb{G}) - \{v_5, v_7\} \cup \{u_5, u_7\}$           |
| $\bar{R}_7 = R_L(v_0, v_7)$    | $V(\mathbb{G}) - \{v_1, v_6\} \cup \{u_1, u_6\}$           |
| $\bar{R}_8 = R_L(v_1, u_1)$    | $V(\mathbb{G})$                                            |
| $\bar{R}_9 = R_L(v_2, u_2)$    | $V(\mathbb{G})$                                            |
| $\bar{R}_{10} = R_L(v_3, u_3)$ | $V(\mathbb{G})$                                            |
| $\bar{R}_{11} = R_L(v_4, u_4)$ | $V(\mathbb{G})$                                            |
| $\bar{R}_{12} = R_L(v_5, u_5)$ | $V(\mathbb{G})$                                            |
| $\bar{R}_{13} = R_L(v_6, u_6)$ | $V(\mathbb{G})$                                            |
| $\bar{R}_{14} = R_L(v_7, u_7)$ | $V(\mathbb{G})$                                            |
| $R_1 = R_L(v_1, v_2)$          | $V(\mathbb{G}) - \{v_4, v_5, v_6\} \cup \{u_4, u_5, u_6\}$ |
| $R_2 = R_L(v_2, v_3)$          | $V(\mathbb{G}) - \{v_7, v_5, v_6\} \cup \{u_7, u_5, u_6\}$ |
| $R_3 = R_L(v_3, v_4)$          | $V(\mathbb{G}) - \{v_1, v_7, v_6\} \cup \{u_1, u_7, u_6\}$ |
| $R_4 = R_L(v_4, v_5)$          | $V(\mathbb{G}) - \{v_1, v_2, v_7\} \cup \{u_1, u_2, u_7\}$ |
| $R_5 R_L(v_5, v_6)$            | $V(\mathbb{G}) - \{v_1, v_2, v_3\} \cup \{u_1, u_2, u_3\}$ |
| $R_6 R_L(v_6, v_7)$            | $V(\mathbb{G}) - \{v_4, v_2, v_3\} \cup \{u_4, u_2, u_3\}$ |
| $R_7 R_L(v_7, v_1)$            | $V(\mathbb{G}) - \{v_3, v_4, v_5\} \cup \{u_3, u_4, u_5\}$ |

Out of the local resolving neighbourhoods given above, we have found that the cardinality for each local resolving neighbourhood  $R_u$  for  $1 \leq u \leq 7$  is 9, which is minimum cardinalities from all other resolving

neighbourhoods  $\bar{R}_u$  for  $1 \leq u \leq 14$ . Moreover,  $\bigcup_{u=1}^7 R_u = V_{\mathbb{G}} - v_0$  so,  $|\bigcup_{u=1}^7 R_u| = 14$  and  $|\bar{R}_l \cap \bigcup_{l=1}^m R_l| \geq |R_l|$

,next we define mapping  $\phi : v(\mathbb{G}) \rightarrow [0, 1]$  such that  $\psi(v_l) = \frac{1}{9}$ , hence, by lemma 5.1.1, we have  
Consequently, We have

$$\dim_{Lfm}(\mathbb{G}) = \sum_{u=1}^{14} \frac{1}{8} < \frac{14}{9} = \frac{14}{9}$$

## Case II

(i) For  $q = 5, m = 13$  and  $1 \leq k \leq 2$  Thus we have,

| RN                                | Elements                                                                                                                                                |
|-----------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| $R_1 = R_L(v_1, v_2)$             | $V(\dot{G}) - \{v_4, v_5, v_6, \dots, v_{12}\} \cup \{u_k^{v_4}, u_k^{v_5}, \dots, u_k^{v_{12}}\}$                                                      |
| $R_2 = R_L(v_2, v_3)$             | $V(\dot{G}) - \{v_5, v_6, \dots, v_{13}\} \cup \{u_k^{v_5}, u_k^{v_6}, \dots, u_k^{v_{13}}\}$                                                           |
| $R_3 = R_L(v_3, v_4)$             | $V(\dot{G}) - \{v_1, v_6, v_7, \dots, v_{13}\} \cup \{u_k^{v_1}, u_k^{v_6}, u_k^{v_7}, \dots, u_k^{v_{13}}\}$                                           |
| $R_4 = R_L(v_4, v_5)$             | $V(\dot{G}) - \{v_1, v_2, v_7, v_8, v_9, \dots, v_{13}\} \cup \{u_1, u_2, u_3, u_4, u_{13}, u_{14}, u_{15}, \dots, u_{26}\}$                            |
| $R_5 = R_L(v_5, v_6)$             | $V(\dot{G}) - \{v_1, v_2, v_3, v_8, v_9, v_{10}, \dots, v_{13}\} \cup \{u_1, u_2, u_3, u_4, u_5, u_6, u_{15}, u_{16}, u_{17}, \dots, u_{26}\}$          |
| $R_6 = R_L(v_6, v_7)$             | $V(\dot{G}) - \{v_1, v_2, v_3, v_4, v_9, v_{10}, v_{11}, v_{12}, v_{13}\} \cup \{u_1, u_2, u_3, \dots, u_8, u_{17}, u_{18}, u_{19}, \dots, u_{26}\}$    |
| $R_7 = R_L(v_7, v_8)$             | $V(\dot{G}) - \{v_1, v_2, v_3, v_4, v_5, v_{10}, v_{11}, v_{12}, v_{13}\} \cup \{u_1, u_2, u_3, \dots, u_{10}, u_{19}, u_{20}, u_{21}, \dots, u_{26}\}$ |
| $R_8 = R_L(v_8, v_9)$             | $V(\dot{G}) - \{v_1, v_2, v_3, v_4, v_5, v_6, v_{11}, v_{12}, v_{13}\} \cup \{u_1, u_2, u_3, \dots, u_{12}, u_{21}, u_{22}, u_{23}, \dots, u_{26}\}$    |
| $R_9 = R_L(v_9, v_{10})$          | $V(\dot{G}) - \{v_1, v_2, v_3, \dots, v_7, v_{12}, v_{13}\} \cup \{u_1, u_2, u_3, \dots, u_{14}, u_{23}, u_{24}, u_{25}, u_{26}\}$                      |
| $R_{10} = R_L(v_{10}, v_{11})$    | $V(\dot{G}) - \{v_1, v_2, v_3, \dots, v_8, v_{13}\} \cup \{u_1, u_2, u_3, \dots, u_{16}, u_{25}, u_{26}\}$                                              |
| $R_{11} = R_L(v_{11}, v_{12})$    | $V(\dot{G}) - \{v_1, v_2, v_3, \dots, v_9, v_{13}\} \cup \{u_1, u_2, u_3, \dots, u_{18}\}$                                                              |
| $R_{12} = R_L(v_{12}, v_{13})$    | $V(\dot{G}) - \{v_2, v_3, v_4, \dots, v_{10}\} \cup \{u_3, u_4, u_5, \dots, u_{20}\}$                                                                   |
| $R_{13} = R_L(v_{13}, v_1)$       | $V(\dot{G}) - \{v_3, v_4, v_5, \dots, v_{11}\} \cup \{u_5, u_6, u_7, \dots, u_{22}\}$                                                                   |
| $\bar{R}_1 = R_L(v_0, v_1)$       | $V(\dot{G}) - \{v_2, v_{13}\} \cup \{u_3, u_4, u_{25}, u_{26}\}$                                                                                        |
| $\bar{R}_2 = R_L(v_0, v_2)$       | $V(\dot{G}) - \{v_3, v_1\} \cup \{u_1, u_2, u_5, u_6\}$                                                                                                 |
| $\bar{R}_3 = R_L(v_0, v_3)$       | $V(\dot{G}) - \{v_2, v_4\} \cup \{u_3, u_4, u_7, u_8\}$                                                                                                 |
| $\bar{R}_4 = R_L(v_0, v_4)$       | $V(\dot{G}) - \{v_3, v_5\} \cup \{u_5, u_6, u_9, u_{10}\}$                                                                                              |
| $\bar{R}_5 = R_L(v_0, v_5)$       | $V(\dot{G}) - \{v_4, v_6\} \cup \{u_7, u_8, u_{11}, u_{12}\}$                                                                                           |
| $\bar{R}_6 = R_L(v_0, v_6)$       | $V(\dot{G}) - \{v_5, v_7\} \cup \{u_9, u_{10}, u_{13}, u_{14}\}$                                                                                        |
| $\bar{R}_7 = R_L(v_0, v_7)$       | $V(\dot{G}) - \{v_6, v_8\} \cup \{u_{11}, u_{12}, u_{15}, u_{16}\}$                                                                                     |
| $\bar{R}_8 = R_L(v_0, v_8)$       | $V(\dot{G}) - \{v_7, v_9\} \cup \{u_3, u_{14}, u_{17}, u_{18}\}$                                                                                        |
| $\bar{R}_9 = R_L(v_0, v_9)$       | $V(\dot{G}) - \{v_8, v_{10}\} \cup \{u_{15}, u_{16}, u_{19}, u_{20}\}$                                                                                  |
| $\bar{R}_{10} = R_L(v_0, v_{10})$ | $V(\dot{G}) - \{v_9, v_{11}\} \cup \{u_{17}, u_{18}, u_{21}, u_{22}\}$                                                                                  |
| $\bar{R}_{11} = R_L(v_0, v_{11})$ | $V(\dot{G}) - \{v_{10}, v_{12}\} \cup \{u_{19}, u_{20}, u_{23}, u_{24}\}$                                                                               |
| $\bar{R}_{12} = R_L(v_0, v_{12})$ | $V(\dot{G}) - \{v_{11}, v_{13}\} \cup \{u_{21}, u_{22}, u_{25}, u_{26}\}$                                                                               |
| $\bar{R}_{13} = R_L(v_0, v_{13})$ | $V(\dot{G}) - \{v_1, v_{12}\} \cup \{u_1, u_2, u_{23}, u_{24}\}$                                                                                        |
| $\bar{R}_{14} = R_L(v_1, u_1)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{15} = R_L(v_1, u_2)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{16} = R_L(v_2, u_3)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{17} = R_L(v_2, u_4)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{18} = R_L(v_3, u_5)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{19} = R_L(v_3, u_6)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{20} = R_L(v_4, u_7)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{21} = R_L(v_4, u_8)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{22} = R_L(v_5, u_9)$    | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{23} = R_L(v_5, u_{10})$ | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{24} = R_L(v_6, u_{11})$ | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{25} = R_L(v_6, u_{12})$ | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{26} = R_L(v_7, u_{13})$ | $V(\dot{G})$                                                                                                                                            |
| $\bar{R}_{27} = R_L(v_7, u_{14})$ | $V(\dot{G})$                                                                                                                                            |

|                                      |              |
|--------------------------------------|--------------|
| $\bar{R}_{28} = R_L(v_8, u_{15})$    | $V(\dot{G})$ |
| $\bar{R}_{29} = R_L(v_8, u_{16})$    | $V(\dot{G})$ |
| $\bar{R}_{30} = R_L(v_9, u_{17})$    | $V(\dot{G})$ |
| $\bar{R}_{31} = R_L(v_9, u_{18})$    | $V(\dot{G})$ |
| $\bar{R}_{32} = R_L(v_{10}, u_{19})$ | $V(\dot{G})$ |
| $\bar{R}_{33} = R_L(v_{10}, u_{20})$ | $V(\dot{G})$ |
| $\bar{R}_{34} = R_L(v_{11}, u_{21})$ | $V(\dot{G})$ |
| $\bar{R}_{35} = R_L(v_{11}, u_{22})$ | $V(\dot{G})$ |
| $\bar{R}_{36} = R_L(v_{12}, u_{23})$ | $V(\dot{G})$ |
| $\bar{R}_{37} = R_L(v_{12}, u_{24})$ | $V(\dot{G})$ |
| $\bar{R}_{38} = R_L(v_{13}, u_{25})$ | $V(\dot{G})$ |
| $\bar{R}_{39} = R_L(v_{13}, u_{26})$ | $V(\dot{G})$ |

From the local resolving neighbourhoods given above, It can be seen that the cardinality for each local resolving neighbourhood  $R_l$  we have for  $1 \leq u \leq 13$  is 13, which is the minimum cardinality among all the other local resolving neighbourhoods  $\bar{R}_u$  for  $1 \leq u \leq 13$  cardinality is 34 and for  $14 \leq l \leq 38$  we have cardinality which is 40. Moreover,  $\bigcup_{l=1}^{13} R_{13} = V_{\mathbb{G}} - v_0$  so,  $|\bigcup_{l=1}^{13} \bar{R}_l \cap \bigcup_{l=1}^{13} R_l| \geq |R_l|$ , next we define a mapping  $\psi : v(\mathbb{G}) \rightarrow [0, 1]$  such that  $\psi(v_l) = \frac{1}{12}$ . Since  $R_l$  for  $1 \leq l \leq 7$  of  $\dot{V}(G)$  therefore, by lemma 5.1.1, we have

$$\dim_{Lfm}(\dot{G}_{II}) = \sum_{l=1}^{39} \frac{1}{13} < \frac{39}{13} < 3$$

(ii) When  $s \geq 5$  The possible local resolving neighbourhoods are  $R_L\{v_0, v_m\}$ ,  $R_L\{v_m, v_{m+1}\}$  and  $R_L\{v_m, u_k^{v_m}\}$ , clearly from the lemma 1  $|R_l| = |R\{v_l, v_{l+1}\}| = 4s - 7 \leq |\{R(uv)\}| \forall E(\dot{G}_{II})$  and  $|\{R(uv)\}| \cap \bigcup_{u=1}^n R_u \geq |R_u|$ . Let

$$\delta = |\bigcup_{u=1}^n R_u| = p^3 - 5p_2 + 9p - 6 \text{ and } \omega = |R_u| = 4p - 7.$$

Now we define a bijection  $\phi : v\mathbb{G} \rightarrow [0, 1]$  such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^n R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^n R_u \end{cases}$$

Now we see that  $\phi$  is a local resolving function for  $\dot{G}_{II}$  because  $\phi(R\{cd\}) \geq 1 \forall cd \in E(\dot{G}_{II})$ . To prove this, suppose on contrary that we have another resolving neighbourhood  $\psi$ , such that  $\psi(c) \leq \phi(c)$  for this at least one  $c \in V(\mathbb{G})$  and  $\phi(c) \neq \psi(c)$ . Consequently, we have  $\phi\{R(cd)\} \geq 1 \forall cd \in E(\mathbb{G})$  Where  $R(cd)$  is the local resolving neighbourhood of  $\mathbb{G}_{II}$  with Cardinality  $\omega$  Which is minimum. From this we see that  $\psi$  is not a local resolving function, so this is a contradiction, thus we can say that  $\phi$  is a resolving function which is minimal for  $\mathbb{G}_{II}$ . Therefore, by lemma 5.1.1, we get:

$$\dim_{\mathbb{G}_{II}} < \sum_{u=1}^{\delta} \frac{1}{\omega} < \frac{s^3 - 5s^2 + 9s - 6}{4s - 7}.$$

**Theorem 5.1.3:** If  $\mathbb{G} \cong G_{II}$ , for  $s \geq 6$  and  $n = s^2 - 3s + 1$ . Then  $\dim_{G_{III}} < \frac{s^3 - 6s^2 + 10s - 3}{4s + 7}$

**Proof: Case 1:** When  $s = 6$  We have the following local resolving neighbourhoods for  $\mathbb{G}$  under consideration.

$$\begin{aligned}
\bar{R}_1 &= R_L(v_0, v_1) = V(\mathbb{G}) - \{v_2, v_{19}\} \cup \{w_2, w_{18}\} \cup \{x_1^{v_2}, x_1^{v_{19}}\} \\
\bar{R}_2 &= R_L(v_0, v_2) = V(\mathbb{G}) - \{v_1, v_3\} \cup \{w_3, w_{19}\} \cup \{x_1^{v_1}, x_1^{v_3}\} \\
\bar{R}_3 &= R_L(v_0, v_3) = V(\mathbb{G}) - \{v_2, v_4\} \cup \{w_1, w_4\} \cup \{x_2^{v_2}, x_1^{v_4}\} \\
\bar{R}_4 &= R_L(v_0, v_4) = V(\mathbb{G}) - \{v_3, v_5\} \cup \{w_2, w_5\} \cup \{x_3^{v_3}, x_1^{v_5}\} \\
\bar{R}_5 &= R_L(v_0, v_5) = V(\mathbb{G}) - \{v_4, v_6\} \cup \{w_3, w_5\} \cup \{x_4^{v_4}, x_1^{v_6}\} \\
\bar{R}_6 &= R_L(v_0, v_6) = V(\mathbb{G}) - \{v_5, v_7\} \cup \{w_4, w_6\} \cup \{x_1^{v_5}, x_1^{v_7}\} \\
\bar{R}_7 &= R_L(v_0, v_7) = V(\mathbb{G}) - \{v_6, v_8\} \cup \{w_5, w_7\} \cup \{x_1^{v_6}, x_1^{v_8}\} \\
\bar{R}_8 &= R_L(v_0, v_8) = V(\mathbb{G}) - \{v_7, v_9\} \cup \{w_6, w_8\} \cup \{x_1^{v_7}, x_1^{v_9}\} \\
\bar{R}_9 &= R_L(v_0, v_9) = V(\mathbb{G}) - \{v_8, v_{10}\} \cup \{w_7, w_9\} \cup \{x_1^{v_8}, x_1^{v_{10}}\} \\
\bar{R}_{10} &= R_L(v_0, v_{10}) = V(\mathbb{G}) - \{v_9, v_{11}\} \cup \{w_8, w_{10}\} \cup \{x_1^{v_9}, x_1^{v_{11}}\} \\
\bar{R}_{11} &= R_L\{v_0, v_{11}\} = V(\mathbb{G}) - \{v_{10}, v_{12}\} \cup \{w_9, w_{11}\} \cup \{x_1^{v_{10}}, x_1^{v_{12}}\} \\
\bar{R}_{12} &= R_L\{v_0, v_{12}\} = V(\mathbb{G}) - \{v_{11}, v_{13}\} \cup \{w_{10}, w_{12}\} \cup \{x_1^{v_{11}}, x_1^{v_{13}}\} \\
\bar{R}_{13} &= R_L\{v_0, v_{13}\} = V(\mathbb{G}) - \{v_{12}, v_{14}\} \cup \{w_{11}, w_{13}\} \cup \{x_1^{v_{12}}, x_1^{v_{14}}\} \\
\bar{R}_{14} &= R_L\{v_0, v_{14}\} = V(\mathbb{G}) - \{v_{13}, v_{15}\} \cup \{w_{12}, w_{14}\} \cup \{x_1^{v_{13}}, x_1^{v_{15}}\} \\
\bar{R}_{15} &= R_L\{v_0, v_{15}\} = V(\mathbb{G}) - \{v_{14}, v_{16}\} \cup \{w_{13}, w_{15}\} \cup \{x_1^{v_{14}}, x_1^{v_{16}}\} \\
\bar{R}_{16} &= R_L\{v_0, v_{16}\} = V(\mathbb{G}) - \{v_{15}, v_{17}\} \cup \{w_{14}, w_{16}\} \cup \{x_1^{v_{15}}, x_1^{v_{17}}\} \\
\bar{R}_{17} &= R_L\{v_0, v_{17}\} = V(\mathbb{G}) - \{v_{16}, v_{18}\} \cup \{w_{15}, w_{17}\} \cup \{x_1^{v_{16}}, x_1^{v_{18}}\} \\
\bar{R}_{18} &= R_L\{v_0, v_{18}\} = V(\mathbb{G}) - \{v_{17}, v_{19}\} \cup \{w_{16}, w_{18}\} \cup \{x_1^{v_{17}}, x_1^{v_{19}}\} \\
\bar{R}_{19} &= R_L\{v_0, v_{19}\} = V(\mathbb{G}) - \{v_{18}, v_{20}\} \cup \{w_{17}, w_{19}\} \cup \{x_1^{v_{18}}, x_1^{v_{20}}\} \quad \bar{R}_{20} = R_L\{v_1, x_1^{v_1}\} = V(\mathbb{G}) \\
\bar{R}_{21} &= R_L\{v_2, x_1^{v_2}\} = V(\mathbb{G}) \quad \bar{R}_{16} = R_L\{v_3, x_1^{v_3}\} = V(\mathbb{G}) \quad \bar{R}_{17} = R_L\{v_4, x_1^{v_4}\} = V(\mathbb{G}) \\
\bar{R}_{18} &= R_L\{v_5, x_1^{v_5}\} = V(\mathbb{G}) \quad \bar{R}_{19} = R_L\{v_6, x_1^{v_6}\} = V(\mathbb{G}) \quad \bar{R}_{20} = R_L\{v_7, x_1^{v_7}\} = V(\mathbb{G}) \\
\bar{R}_{21} &= R_L\{v_8, x_1^{v_8}\} = V(\mathbb{G}) \quad \bar{R}_{22} = R_L\{v_9, x_1^{v_9}\} = V(\mathbb{G}) \quad \bar{R}_{23} = R_L\{v_{10}, x_1^{v_{10}}\} = V(\mathbb{G}) \\
\bar{R}_{24} &= R_L\{v_{11}, x_1^{v_{11}}\} = V(\mathbb{G}) \quad \bar{R}_{25} = R_L\{v_{12}, x_1^{v_{12}}\} = V(\mathbb{G}) \quad \bar{R}_{26} = R_L\{v_{13}, x_1^{v_{13}}\} = V(\mathbb{G}) \\
\bar{R}_{27} &= R_L\{v_{14}, x_1^{v_{14}}\} = V(\mathbb{G}) \quad \bar{R}_{28} = R_L\{v_{15}, x_1^{v_{15}}\} = V(\mathbb{G}) \quad \bar{R}_{29} = R_L\{v_{16}, x_1^{v_{16}}\} = V(\mathbb{G}) \\
\bar{R}_{30} &= R_L\{v_{17}, x_1^{v_{17}}\} = V(\mathbb{G}) \quad \bar{R}_{31} = R_L\{v_{18}, x_1^{v_{18}}\} = V(\mathbb{G}) \quad \bar{R}_{32} = R_L\{v_{19}, x_1^{v_{19}}\} = V(\mathbb{G}) \\
\bar{R}_{33} &= R_L\{v_1, w_1\} = V(\mathbb{G}) - \{v_2, v_3\} \cup \{w_2, w_3\} \cup \{x_1^{v_2}, x_1^{v_3}\} \\
\bar{R}_{34} &= R_L\{v_2, w_2\} = V(\mathbb{G}) - \{v_4, v_5\} \cup \{w_4, w_5\} \cup \{x_1^{v_4}, x_1^{v_5}\} \\
&\vdots \\
&\vdots \\
\bar{R}_{50} &= R_L\{v_{18}, w_{18}\} = V(\mathbb{G}) - \{v_1, v_{19}\} \cup \{w_1, w_{19}\} \cup \{x_1^{v_1}, x_1^{v_{19}}\} \\
\bar{R}_{51} &= R_L\{v_{12}, w_{23}\} = V(\mathbb{G}) - \{v_1, v_2\} \cup \{w_1, w_2\} \cup \{x_1^{v_1}, x_1^{v_2}\} \\
\bar{R}_{52} &= R_L\{v_2, w_1\} = V(\mathbb{G}) - \{v_1, v_{19}\} \cup \{w_1, w_{19}\} \cup \{x_1^{v_1}, x_1^{v_{19}}\} \\
\bar{R}_{53} &= R_L\{v_3, w_2\} = V(\mathbb{G}) - \{v_1, v_2\} \cup \{w_1, w_2\} \cup \{x_1^{v_1}, x_1^{v_2}\} \vdots \\
\bar{R}_{69} &= R_L\{v_{19}, w_{18}\} = V(\mathbb{G}) - \{v_{17}, v_{18}\} \cup \{w_{17}, w_{18}\} \cup \{x_1^{v_{17}}, x_1^{v_{18}}\} \\
\bar{R}_{70} &= R_L\{v_1, w_{19}\} = V(\mathbb{G}) - \{v_{18}, v_{19}\} \cup \{w_{18}, w_{19}\} \cup \{x_1^{v_{18}}, x_1^{v_{19}}\} \\
R_1 &= R_L\{v_1, v_2\} = V(\mathbb{G}) - \{v_4, v_5, v_6, \dots, v_{18}\} \cup \{w_1, w_4, w_5, w_6, w_7, \dots, w_{17}\} \cup \{x_4, x_5, x_6, \dots, x_{17}\} \\
R_2 &= R_L\{v_2, v_3\} = V(\mathbb{G}) - \{v_5, v_6, v_7, \dots, v_{19}\} \cup \{w_2, w_5, w_6, w_7, \dots, w_{18}\} \cup \{x_5, x_6, x_7, \dots, x_{18}\} \\
R_3 &= R_L\{v_3, v_4\} = V(\mathbb{G}) - \{v_1, v_6, v_7, v_8, \dots, v_{19}\} \cup \{w_3, w_6, w_7, w_8, \dots, w_{19}\} \cup \{x_6, x_7, x_8, \dots, x_{19}\} \quad R_4 = \\
&R_L\{v_4, v_5\} = V(\mathbb{G}) - \{v_1, v_2, v_7, v_8, v_9, \dots, v_{19}\} \cup \{w_1, w_4, w_7, w_8, w_9, \dots, w_{19}\} \cup \{x_1, x_7, x_8, x_9, \dots, x_{19}\} \\
R_5 &= R_L\{v_5, v_6\} = \\
&V(\mathbb{G}) - \{v_1, v_2, v_3, v_8, v_9, v_{10}, \dots, v_{19}\} \cup \{w_1, w_2, w_5, w_8, w_9, w_{10}, \dots, w_{19}\} \cup \{x_1, x_2, x_8, x_9, x_{10}, \dots, x_{19}\} \\
R_6 &= R_L\{v_6, v_7\} = V(\mathbb{G}) - \{v_1, v_2, v_3, v_4, v_9, v_{10}, v_{11}, \dots, v_{19}\} \cup \{w_1, w_2, w_3, w_6, w_9, w_{10}, w_{11}, \dots, w_{19}\} \cup \\
&\{x_1, x_2, x_3, x_9, x_{10}, x_{11}, \dots, x_{19}\} \quad R_7 = R_L\{v_7, v_8\} = V(\mathbb{G}) - \{v_1, v_2, v_3, v_4, v_5, v_{10}, v_{11}, v_{12}, \dots, v_{19}\} \cup \\
&\{w_1, w_2, w_3, w_4, w_7, w_{10}, w_{11}, w_{12}, \dots, w_{19}\} \cup \{x_1, x_2, x_3, x_4, x_{10}, x_{11}, x_{12}, \dots, x_{19}\} \\
R_8 &= R_L\{v_8, v_9\} = V(\mathbb{G}) - \{v_1, v_2, v_3, v_4, v_5, v_6, v_{11}, v_{12}, v_{13}, \dots, v_{19}\} \cup \\
&\{w_1, w_2, w_3, \dots, w_5, w_8, w_{11}, w_{12}, w_{13}, \dots, w_{19}\} \cup \{x_1, x_2, x_3, x_4, x_5, x_{11}, x_{12}, x_{13}, \dots, x_{19}\} \\
R_9 &= R_L\{v_9, v_{10}\} = V(\mathbb{G}) - \{v_1, v_2, v_3, \dots, v_7, v_{12}, v_{13}, v_{14}, \dots, v_{19}\} \cup \\
&\{w_1, w_2, w_3, \dots, w_6, w_9, w_{12}, w_{13}, w_{14}, \dots, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_6, x_{12}, x_{13}, x_{14}, \dots, x_{19}\} \\
R_{10} &= R_L\{v_{10}, v_{11}\} = V(\mathbb{G}) - \{v_1, v_2, v_3, \dots, v_8, v_{13}, v_{14}, v_{15}, \dots, v_{19}\} \cup
\end{aligned}$$

$$\begin{aligned}
& \{w_1, w_2, w_3, \dots, w_7, w_{10}, w_{13}, w_{14}, w_{15}, \dots, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_7, x_{12}, x_{13}, x_{14}, \dots, x_{19}\} \\
R_{11} &= R_L\{v_{11}, v_{12}\} = V(\mathbb{G}) - \{v_1 v_2, v_3, \dots, v_9, v_{14}, v_{15}, v_{16}, \dots, v_{19}\} \cup \\
& \{w_1, w_2, w_3, \dots, w_8, w_{11}, w_{14}, w_{15}, w_{16}, \dots, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_8, w_{14}x_{15}, x_{16}, \dots, x_{19}\} \\
R_{12} &= R_L\{v_{12}, v_{13}\} = V(\mathbb{G}) - \{v_1, v_2, v_3, \dots, v_{10}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}\} \cup \\
& \{w_1, w_2, w_3, \dots, w_9, w_{12}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_9, x_{15}x_{16}, x_{17}, x_{18}, x_{19}\} \\
R_{13} &= R_L\{v_{13}, v_{14}\} = V(\mathbb{G}) - \{v_1, v_2, v_3, \dots, v_{11}, v_{16}, v_{17}, v_{18}, v_{19}\} \cup \\
& \{w_1, w_2, w_3, \dots, w_{10}, w_{13}, w_{16}, w_{17}, w_{18}, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_{10}, x_{16}, x_{17}, x_{18}, x_{19}\} \\
R_{14} &= R_L\{v_{14}, v_{15}\} = V(\mathbb{G}) - \{v_1, v_2, v_3, \dots, v_{12}, v_{17}, v_{18}, v_{19}\} \cup \\
& \{w_1, w_2, w_3, \dots, w_{11}, w_{14}, w_{17}, w_{18}, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_{11}, x_{17}, x_{18}, x_{19}\} \\
R_{15} &= R_L\{v_{15}, v_{16}\} = V(\mathbb{G}) - \\
& \{v_1, v_2, v_3, \dots, v_{13}, v_{18}, v_{19}\} \cup \{w_1, w_2, w_3, \dots, w_{12}, w_{15}, w_{18}, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_{10}, x_{16}, x_{17}, x_{18}, x_{19}\} \\
R_{16} &= R_L\{v_{16}, v_{17}\} = \\
& V(\mathbb{G}) - \{v_1 v_2, v_3, \dots, v_{14}, v_{19}\} \cup \{w_1, w_2, w_3, \dots, w_{13}, w_{16}, w_{19}\} \cup \{x_1, x_2, x_3, \dots, x_{13}, x_{19}\} \\
R_{17} &= R_L\{v_{17}, v_{18}\} = V(\mathbb{G}) - \{v_1 v_2, v_3, \dots, v_{15}\} \cup \{w_1, w_2, w_3, \dots, w_{14}, w_{17}\} \cup \{x_1, x_2, x_3, \dots, x_{14}, w_{14}\} \\
R_{18} &= R_L\{v_{18}, v_{19}\} = V(\mathbb{G}) - \{v_2, v_3, v_4, \dots, v_{16}\} \cup \{w_2, w_3, \dots, w_{14}, w_{18}\} \cup \{x_1, x_2, x_3, \dots, x_{14}\} \\
R_{19} &= R_L\{v_{19}, v_1\} = V(\mathbb{G}) - \{v_3, v_4, v_5, \dots, v_{17}\} \cup \{w_3, w_4, w_5, \dots, w_{15}, w_{19}\} \cup \{x_2, x_3, x_4, \dots, x_{15}\}
\end{aligned}$$

From the above resolving neighbourhood, we see that ,the cardinality for each local resolving neighbourhood  $R_l$  we have for  $1 \leq l \leq 19$  is 31, which is the minimum cardinalities from all other resolving neighbourhoods  $\bar{R}_l$  for  $1 \leq l \leq 19$  cardinality is 51 and for  $20 \leq l \leq 57$  we have cardinality which is 58 and we have for  $58 \leq l \leq 95$  is 52. Moreover,  $\bigcup_{l=1}^{19} R_{19} = V_{\mathbb{G}} - v_0$  so,  $|\bigcup_{l=1}^{13} R_l| = 39$  and  $|\bar{R}_l \cap \bigcup_{l=1}^{13} R_l| \geq |R_l|$  ,next we define function or mapping  $\psi : v(\mathbb{G}) \rightarrow [0, 1]$  such that  $\psi(v_l) = \frac{1}{14}$  , thus by lemma 5.1.2, Consequently, We have

$$dim_{Lfm}(\acute{G}_{III}) = \sum_{l=1}^{57} \frac{1}{31} < \frac{57}{31}$$

**Case-2:** for  $p \geq 6$ ,  $m = p^2 - 3p + 1$  thus,we have The possible local resolving neighbourhoods are  $R_L(v_0, v_l), R_L(v_l, v_{l+1})$ ,  $R_L(u_l, v_j)$ , and  $R_L(v_l, w_j)$  clearly from the lemma 1  $|R_l| = |R\{v_l, v_{l+1}\}| = p^3 - 5p^2 + 9p - 9 \leq |\{R(uv)\}|$  now  $\forall E(\acute{G}_{II})$  and  $|\{R(uv)\}| \cap \bigcup_{l=1}^m R_l \geq |R_l|$  Let  $\gamma = |\bigcup_{l=1}^m R_l| = p^3 - 5p^2 + 9p - 6$  and  $\beta = |R_l| = |R_l| = p^3 - 5p^2 + 9p - 9$  Now we define a function or mapping  $\phi : v\mathbb{G} \rightarrow [0, 1]$  such that

$$\psi(v) = \begin{cases} \frac{1}{\beta} & \text{for } v \in \bigcup_{l=1}^m R_l \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{l=1}^m R_l \end{cases}$$

Now we see that  $\psi$  is local resolving function for  $\acute{G}_{III}$  because  $\psi(R\{uv\}) \geq 1 \forall uv \in E(\acute{G}_{II})$ . To prove this ,suppose on contrary that we have another resolving neighbourhood  $\phi$ , such that  $\phi(u) \leq \psi(u)$  for this at least one  $u \in v(\acute{G}_{II})$  and  $\phi(u) \neq \psi(u)$ . Consequently, we have  $\phi\{R(uv)\} \geq 1 \forall uv \in E(\acute{G}_{III})$  Where  $R(uv)$  is the local resolving neighbourhood of  $G_{III}$  with Cardinality  $\beta$  Which is minimum .From this we see that  $\phi$  is not s local resolving function,so this is contradiction,thus we can say that  $\psi$  is a resolving function which is minimal for  $G_{III}$ . By lemma 5.1.2, we assigning  $\frac{1}{\beta}$  to all vertices of  $G_{III}$  except  $v_0$  and sum up all these weights we get:  $dim_{G_{III}} < \sum_{i=1}^{\gamma} < \frac{s^3 - 6s^2 + 10s - 3}{4s + 7}$ .

### 5.1. Local Fractional Metric Dimension of $GDW(m; q_1, q_2, \dots, q_m)$

**Theorem :**(a) If  $\mathbb{G} \cong DW(p, q)$ , for any  $p \geq 3$ , and  $q \equiv 0(mod 2)$ . Then  $dim_{lfac}(\mathbb{G}) = 1$ .  
(b) If  $\mathbb{G} \cong DW(p, q)$ , for any  $p \geq 3$ , and  $q \equiv 1(mod 2)$ . Then  $dim_{lfac}(\mathbb{G}) = p$ .

**Proof:**(a) Since the graph so obtained will be bipartite, thus

$$\dim_{lfrac}(\mathbb{G}) = 1.$$

**Theorem 5.2.1:**(a) If  $\mathbb{G} \cong GDW(m; q_1, q_2, \dots, q_m)$ , for any  $m, n \geq 3$ ,  $1 \leq p \leq m$ ,  $n \equiv 0(mod 2)$  and  $q_p = n$ . Then

$$\dim_{lfrac}(\mathbb{G}) = 1.$$

(b) If  $\mathbb{G} \cong GDW(m; q_1, q_2, \dots, q_m)$ , for any  $m, n \geq 3$ ,  $1 \leq i \leq m$ ,  $n \equiv 1(mod 2)$  and  $q_i = n$ . Then  $\dim_{lfrac}(\mathbb{G}) = m$ .

**Proof:**(a) Since the graph so obtained will be bipartite, thus

$$\dim_{lfrac}(\mathbb{G}) = 1.$$

(b)**Case 2:**

(i) When  $m = 3$  and  $q = 5$

The local resolving neighbourhoods are given by

$$\begin{aligned} R_1 &= R\{x_2^1 x_3^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3\}, \\ R_2 &= R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1, x_4^1\} \cup \{x_1^3, x_2^3\}, \\ R_3 &= R\{x_2^3 x_3^3\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1, x_4^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3\}, \\ R_5 &= R\{vx_1^1\} = V(\mathbb{G}) - \{x_3^1\}, R_6 = R\{vx_1^2\} = V(\mathbb{G}) - \{x_2^2\}, R_8 = R\{vx_4^2\} = V(\mathbb{G}) - \{x_2^2\}, \\ R_7 &= R\{vx_1^3\} = V(\mathbb{G}) - \{x_3^3\}, R_8 = R\{vx_4^1\} = V(\mathbb{G}) - \{x_2^1\}, \\ R_9 &= R\{vx_4^2\} = V(\mathbb{G}) - \{x_2^2\}, R_9 = R\{vx_4^3\} = V(\mathbb{G}) - \{x_2^3\}, \\ R_{10} &= R\{x_1^1 x_2^1\} = V(\mathbb{G}) - \{x_4^1\}, R_{11} = R\{x_1^2 x_2^2\} = V(\mathbb{G}) - \{x_4^2\} \\ R_{12} &= R\{x_1^3 x_2^3\} = V(\mathbb{G}) - \{x_4^3\}, R_{13} = R\{x_3^1 x_4^1\} = V(\mathbb{G}) - \{x_1^1\}, \\ R_{14} &= R\{x_3^2 x_4^2\} = V(\mathbb{G}) - \{x_1^2\}, R_{15} = R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{x_1^3\}. \end{aligned}$$

Among all the local resolving neighbourhoods, for  $1 \leq u \leq 3$ ,  $R_1$  is having the minimum cardinality of 4. Whereas the rest of them for  $4 \leq u \leq 9$  are having the maximum cardinality that is 12. Thus

$\bigcup_{u=1}^3 R_u = V(\mathbb{G}) - \{v\}$  and  $|\bigcup_{u=1}^3 R_u| = 12$ . Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that assigning  $\frac{1}{4}$  to all the vertices of  $\mathbb{G}$  except  $\{v\}$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^p \frac{1}{4} = \frac{12}{4} = 3.$$

(ii) When  $m = 4$  and  $q = 5$

The local resolving neighbourhoods are given by

$$\begin{aligned} R_1 &= R\{x_2^1 x_3^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4\}, \\ R_2 &= R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1, x_4^1\} \cup \{x_1^3, x_2^3\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4\}, \\ R_3 &= R\{x_2^3 x_3^3\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1, x_4^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3\}, \\ R_4 &= R\{x_2^4 x_3^4\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1, x_4^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3\}, \\ R_5 &= R\{vx_1^1\} = V(\mathbb{G}) - \{x_3^1\}, R_6 = R\{vx_4^1\} = V(\mathbb{G}) - \{x_2^1\}, \\ R_7 &= R\{vx_1^2\} = V(\mathbb{G}) - \{x_3^2\}, R_8 = R\{vx_4^2\} = V(\mathbb{G}) - \{x_2^2\}, \\ R_9 &= R\{vx_1^3\} = V(\mathbb{G}) - \{x_3^3\}, R_{10} = R\{vx_4^3\} = V(\mathbb{G}) - \{x_2^3\}. \\ R_{11} &= R\{vx_1^4\} = V(\mathbb{G}) - \{x_3^4\}, R_{12} = R\{vx_4^4\} = V(\mathbb{G}) - \{x_2^4\}. \end{aligned}$$

$$\begin{aligned}
R_{13} &= R\{x_1^1 x_2^1\} = V(\mathbb{G}) - \{x_4^1\}, R_{14} = R\{x_1^2 x_2^2\} = V(\mathbb{G}) - \{x_4^2\} \\
R_{15} &= R\{x_1^3 x_2^3\} = V(\mathbb{G}) - \{x_4^3\}, R_{16} = R\{x_1^4 x_2^4\} = V(\mathbb{G}) - \{x_4^4\}, \\
R_{17} &= R\{x_3^1 x_4^1\} = V(\mathbb{G}) - \{x_1^1\}, R_{18} = R\{x_3^2 x_4^2\} = V(\mathbb{G}) - \{x_1^2\} \\
R_{19} &= R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{x_1^3\}, R_{20} = R\{x_3^4 x_4^4\} = V(\mathbb{G}) - \{x_1^4\},
\end{aligned}$$

From the local resolving neighbourhoods given above, for  $1 \leq u \leq 4$ ,  $R_1$  is having the minimum cardinality of 4. Whereas the rest of them for  $4 \leq u \leq 20$  are having the maximum cardinality that is 16. Thus  $\bigcup_{u=1}^3 R_u = V(\mathbb{G}) - \{v\}$  and  $|\bigcup_{u=1}^3 R_u| = 16$ . Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  such that assigning  $\frac{1}{4}$  to all the vertices of  $\mathbb{G}$  except  $\{v\}$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^p \frac{1}{4} = \frac{16}{4} = 4.$$

(iii) When  $m \geq 4$ ,  $1 \leq p \leq m$  and  $q_p = n$ .

The requisite local resolving neighbourhoods are for  $1 \leq j \leq q_p - 1$ ,  $x_j^p x_{j+1}^p$  and  $vx_j^p$ . Among all the local resolving neighbourhoods, the sets with minimum cardinality are  $x_r^p x_s^p$  having the cardinality of  $n - 1$ . As we have seen in lemma 4.3.1 that  $\bigcup_{u=1}^m R_u = V(\mathbb{G}) - 1$  and  $|\bigcup_{u=1}^m R_u| = mn - 1$ . Suppose  $\delta = |\bigcup_{u=1}^m R_u|$  and  $|R_u| = \omega$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

Thereby assigning the label  $\frac{1}{\omega}$  to all the vertices of  $\mathbb{G}$  except  $\{v\}$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(V(\mathbb{G})) = \sum_{u=1}^{\delta} \frac{1}{\omega} = \frac{m(n-1)}{n-1} = m$$

**Theorem 5.2.2:** If  $\mathbb{G} \cong GDW(m; q_1, q_2, \dots, q_m)$ , For  $m, n \geq 3$ ,  $1 \leq p \leq m$  and  $q_i = n + p - 1$ . Then  $\dim_{lfrac}(\mathbb{G}) = 1$ .

**Proof:**

**Case 1:** If  $n \equiv 1 \pmod{2}$ .

(i) When  $m = 3$  and  $q_1 = 3, q_2 = 4$  and  $q_3 = 5$ .

The local resolving neighbourhoods are given by

$$\begin{aligned}
R_1 &= R\{x_1^1 x_2^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^2, x_2^2, x_3^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}, \\
R_2 &= R\{x_3^1 x_4^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\}, \\
R_3 &= R\{x_1^3 x_2^3\} = V(\mathbb{G}) - \{x_5^3\}, R_4 = R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{x_6^3\} \\
R_5 &= R\{x_4^3 x_5^3\} = V(\mathbb{G}) - \{x_1^3\}, R_6 = R\{x_5^3 x_6^3\} = V(\mathbb{G}) - \{x_2^3\} \\
R_7 &= R\{vx_1^1\} = V(\mathbb{G}) - \{x_3^1\}, R_8 = R\{vx_3^1\} = V(\mathbb{G}) - \{x_2^1\}, \\
R_9 &= R\{vx_1^3\} = V(\mathbb{G}) - \{x_4^3\}, R_{10} = R\{vx_3^3\} = V(\mathbb{G}) - \{x_5^3\}. \\
R_{11} &= R\{x_1^2 x_2^2\} = V(\mathbb{G}), R_{12} = R\{x_2^2 x_3^2\} = V(\mathbb{G}), \\
R_{13} &= R\{vx_1^2\} = V(\mathbb{G}), R_{14} = R\{vx_3^2\} = V(\mathbb{G}).
\end{aligned}$$

Among all the local resolving neighbourhoods,  $R_1$  is having the minimum cardinality of 2. Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  thereby assigning  $\frac{1}{2}$  to all the vertices in the set  $R_1$  and summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^2 \frac{1}{2} = \frac{2}{2} = 1.$$

Notice that equality holds as there is only one and unique minimum local resolving neighbourhood.

(ii) When  $m = 4$  and  $q_1 = 3, q_2 = 4, q_3 = 5$  and  $q_4 = 6$ .

The local resolving neighbourhoods are given by

$$R_1 = R\{x_1^1 x_2^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^2, x_2^2, x_3^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4\},$$

$$R_2 = R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2, x_3^2\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4\},$$

$$R_3 = R\{x_1^3 x_2^3\} = V(\mathbb{G}) - \{x_5^3\}, R_4 = R\{x_2^3 x_3^3\} = V(\mathbb{G}) - \{x_6^3\},$$

$$R_5 = R\{x_4^3 x_5^3\} = V(\mathbb{G}) - \{x_1^3\}, R_6 = R\{x_5^3 x_6^3\} = V(\mathbb{G}) - \{x_2^3\},$$

$$R_7 = R\{vx_1^1\} = V(\mathbb{G}) - \{x_2^1\}, R_8 = R\{vx_1^3\} = V(\mathbb{G}) - \{x_4^1\},$$

$$R_9 = R\{vx_2^1\} = V(\mathbb{G}) - \{x_1^1\}, R_{10} = R\{vx_6^3\} = V(\mathbb{G}) - \{x_1^3\},$$

$$R_{11} = R\{x_1^2 x_2^2\} = V(\mathbb{G}), R_{12} = R\{x_2^2 x_3^2\} = V(\mathbb{G}),$$

$$R_{13} = R\{x_1^4 x_2^4\} = V(\mathbb{G}), R_{14} = R\{x_2^4 x_3^4\} = V(\mathbb{G}),$$

$$R_{15} = R\{x_4^4 x_5^4\} = V(\mathbb{G}), R_{16} = R\{vx_1^2\} = V(\mathbb{G}),$$

$$R_{17} = R\{vx_3^2\} = V(\mathbb{G}), R_{18} = R\{vx_4^4\} = V(\mathbb{G}),$$

$R_{19} = R\{vx_5^4\} = V(\mathbb{G})$ . From all the local resolving neighbourhoods, It is found that  $R_1$  is having the minimum cardinality of 2. So, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  thereby assigning  $\frac{1}{2}$  to all the vertices in the set  $R_1$  and and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^2 \frac{1}{2} = \frac{2}{2} = 1.$$

Notice that equality holds as there is only one and unique minimum local resolving neighbourhood.

**Case 2:** If  $n \equiv 0(mod 2)$ .

(i) When  $m = 3$  and  $q_1 = 4, q_2 = 5$  and  $q_3 = 6$ .

The local resolving neighbourhoods are given by

$$R_1 = R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3\},$$

$$R_2 = R\{x_1^2 x_2^2\} = \{x_4^4\}, R_3 = R\{x_3^2 x_4^2\} = V(\mathbb{G}) - \{x_1^2\},$$

$$R_4 = R\{x_1^3 x_2^3\} = V(\mathbb{G}), R_5 = R\{x_2^3 x_3^3\} = V(\mathbb{G})$$

$$R_6 = R\{x_3^3 x_4^3\} = V(\mathbb{G}), R_7 = R\{x_1^1 x_2^1\} = V(\mathbb{G}),$$

$$R_8 = R\{x_2^1 x_3^1\} = V(\mathbb{G}), R_9 = R\{vx_1^1\} = V(\mathbb{G}) - \{x_3^1\},$$

$$R_{10} = R\{vx_3^1\} = V(\mathbb{G}) - \{x_2^1\}, R_{11} = R\{vx_1^3\} = V(\mathbb{G}) - \{x_4^3\},$$

$$R_{12} = R\{vx_6^3\} = V(\mathbb{G}) - \{x_3^3\}, R_{13} = R\{x_1^2 x_2^2\} = V(\mathbb{G}),$$

$$R_{14} = R\{x_2^2 x_3^2\} = V(\mathbb{G}), R_{15} = R\{vx_1^2\} = V(\mathbb{G}) - \{x_3^2\},$$

$$R_{16} = R\{vx_4^2\} = V(\mathbb{G}) - \{x_1^2\}.$$

Among all the local resolving neighbourhoods,  $R_1$  is having the minimum cardinality of 4. Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  thereby assigning  $\frac{1}{4}$  to all the vertices in the set  $R_1$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^4 \frac{1}{4} = \frac{4}{4} = 1.$$

(ii) When  $m = 4$  and  $q_1 = 4, q_2 = 5, q_3 = 6$  and  $q_4 = 7$ .

The local resolving neighbourhoods are given by

$$R_1 = R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\},$$

$$R_2 = R\{x_3^4 x_4^4\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1, x_3^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3\},$$

$$R_2 = R\{x_1^2 x_2^2\} = V(\mathbb{G}) - \{x_4^4\}, R_3 = R\{x_3^2 x_4^2\} = V(\mathbb{G}) - \{x_1^2\},$$

$$R_4 = R\{x_1^4 x_2^4\} = V(\mathbb{G}) - \{x_5^4\}, R_4 = R\{x_2^4 x_3^4\} = V(\mathbb{G}) - \{x_6^4\},$$

$$R_4 = R\{x_4^4 x_5^4\} = V(\mathbb{G}) - \{x_1^4\}, R_4 = R\{x_5^4 x_6^4\} = V(\mathbb{G}) - \{x_2^4\},$$

$$\begin{aligned}
R_3 &= R\{x_1^3 x_2^3\} = V(\mathbb{G}), R_4 = R\{x_2^3 x_3^3\} = V(\mathbb{G}) \\
R_5 &= R\{x_3^3 x_4^3\} = V(\mathbb{G}), R_6 = R\{x_4^3 x_5^3\} = V(\mathbb{G}) \\
R_7 &= R\{x_1^1 x_2^1\} = V(\mathbb{G}), R_8 = R\{x_2^1 x_3^1\} = V(\mathbb{G}) \\
R_9 &= R\{vx_1^1\} = V(\mathbb{G}) - \{x_3^1\}, R_{10} = R\{vx_3^1\} = V(\mathbb{G}) - \{x_2^1\}, \\
R_{11} &= R\{vx_1^3\} = V(\mathbb{G}) - \{x_4^3\}, R_{12} = R\{vx_6^3\} = V(\mathbb{G}) - \{x_3^3\}. \\
R_{13} &= R\{x_1^2 x_2^2\} = V(\mathbb{G}), R_{14} = R\{x_2^2 x_3^2\} = V(\mathbb{G}), \\
R_{15} &= R\{vx_1^2\} = V(\mathbb{G}) - \{x_3^2\}, R_{16} = R\{vx_4^2\} = V(\mathbb{G}) - \{x_1^2\}. \\
R_{15} &= R\{vx_1^4\} = V(\mathbb{G}) - \{x_4^2\}, R_{16} = R\{vx_6^2\} = V(\mathbb{G}) - \{x_3^2\}.
\end{aligned}$$

Among all the local resolving neighbourhoods given above, we have found that  $R_1$  is having the minimum cardinality of 4. So, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  thereby assigning  $\frac{1}{4}$  to all the vertices in the set  $R_1$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^4 \frac{1}{2} = \frac{4}{4} = 1.$$

**Case III:** When  $m \geq 3$ ,  $1 \leq p \leq m$  and  $q_p = n + p - 1$ .

The required local resolving neighbourhoods are for  $1 \leq j \leq q_p - 1$ ,  $x_j^p x_{j+1}^p$  and  $vx_j^p$ . The one with minimum cardinality will be  $x_{r_\alpha}^\alpha x_{s_\alpha}^\alpha$ , where  $\alpha = \min\{q_p | 1 \leq p \leq m\}$  and  $\alpha \equiv 1 \pmod{2}$ . This means that the minimum local resolving neighbourhood will be one and unique with  $R_1 = \alpha - 1$ . As we have seen in lemma 4.3.1 that  $\bigcup_{u=1}^m R_u = \alpha - 1$  and  $|\bigcup_{u=1}^m R_u| = \alpha - 1$ . Suppose  $\delta = |\bigcup_{u=1}^m R_u| = |R_u| = \omega$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

Thereby assigning the label  $\frac{1}{\omega}$  to all the vertices of  $\alpha$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(V(\mathbb{G})) = \sum_{u=1}^{\omega} \frac{1}{\omega} = \frac{\alpha-1}{\alpha-1} = 1$$

**Theorem 5.2.3:**(a) If  $\mathbb{G} \cong GDW(m; q_1, q_2, \dots, q_m)$ , For  $m, n \geq 3$ ,  $1 \leq p \leq m$  and  $q_p = n + 2(p - 1)$ . Then  $\dim_{lfrac}(\mathbb{G}) = 1$ .

**Proof: Case 1:** If  $n \equiv 0 \pmod{2}$ .

Since the graph so obtained is bipartite as all the  $q_i$  will be the cycle of even length, thus

$$\dim_{lfrac}(\mathbb{G}) = 1.$$

(b) **Case 2:** If  $n \equiv 1 \pmod{2}$ .

(i) When  $m = 3$  and  $q_1 = 3, q_2 = 5$  and  $q_3 = 7$ .

The local resolving neighbourhoods are given by

$$\begin{aligned}
R_1 &= R\{x_1^1 x_2^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\} \\
R_2 &= R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}, \\
R_3 &= R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\}, \\
R_4 &= R\{x_1^2 x_2^2\} = V(\mathbb{G}) - \{x_4^2\}, R_5 = R\{x_1^3 x_2^3\} = V(\mathbb{G}) - \{x_5^3\}, \\
R_6 &= R\{x_2^3 x_3^3\} = V(\mathbb{G}) - \{x_6^3\}, R_7 = R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{x_1^2\}, \\
R_8 &= R\{x_3^2 x_4^2\} = V(\mathbb{G}) - \{x_1^2\}, R_9 = R\{x_4^3 x_5^3\} = V(\mathbb{G}) - \{x_1^3\}, \\
R_{10} &= R\{x_5^3 x_6^3\} = V(\mathbb{G}) - \{x_2^3\}, R_{11} = R\{vx_1^1\} = V(\mathbb{G}) - \{x_3^1\},
\end{aligned}$$

$$R_{12} = R\{vx_3^1\} = V(\mathbb{G}) - \{x_2^1\}, R_{13} = R\{vx_1^2\} = V(\mathbb{G}) - \{x_3^2\}, \\ R_{14} = R\{vx_1^3\} = V(\mathbb{G}) - \{x_4^3\}, R_{15} = R\{vx_6^3\} = V(\mathbb{G}) - \{x_3^3\}.$$

Among all the local resolving neighbourhoods,  $R_1$  is having the minimum cardinality of 2. We define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  thereby assigning  $\frac{1}{2}$  to all the vertices in the set  $R_1$  and summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^2 \frac{1}{2} = \frac{2}{2} = 1.$$

Notice that equality holds as there is only one and unique minimum local resolving neighbourhood.

(ii) When  $m = 3$  and  $q_1 = 3, q_2 = 5, q_3 = 7$  and  $q_4 = 9$ .

The local resolving neighbourhoods are given by

$$R_1 = R\{x_1^1 x_2^1\} = V(\mathbb{G}) - \{v\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4\}, \\ R_2 = R\{x_2^2 x_3^2\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4\}, \\ R_3 = R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4\}, \\ R_4 = R\{x_4^4 x_5^4\} = V(\mathbb{G}) - \{v\} \cup \{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2, x_3^2, x_4^2\} \cup \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}, \\ R_5 = R\{x_1^2 x_2^2\} = V(\mathbb{G}) - \{x_4^2\}, R_6 = R\{x_1^3 x_2^3\} = V(\mathbb{G}) - \{x_5^3\}, \\ R_7 = R\{x_1^4 x_2^4\} = V(\mathbb{G}) - \{x_6^4\}, R_8 = R\{x_3^3 x_4^3\} = V(\mathbb{G}) - \{x_6^3\}, \\ R_9 = R\{x_4^4 x_5^4\} = V(\mathbb{G}) - \{x_7^4\}, R_{10} = R\{x_3^2 x_4^2\} = V(\mathbb{G}) - \{x_1^2\}, R_{11} = R\{x_3^4 x_4^4\} = V(\mathbb{G}) - \{x_8^4\}, \\ R_{12} = R\{x_4^3 x_5^3\} = V(\mathbb{G}) - \{x_1^3\}, R_{13} = R\{x_5^3 x_6^3\} = V(\mathbb{G}) - \{x_2^3\}, \\ R_{14} = R\{x_5^4 x_6^4\} = V(\mathbb{G}) - \{x_1^4\}, R_{15} = R\{x_6^4 x_7^4\} = V(\mathbb{G}) - \{x_2^4\}, R_{16} = R\{x_7^4 x_8^4\} = V(\mathbb{G}) - \{x_4^4\}, R_{17} = R\{vx_1^1\} = V(\mathbb{G}) - \{x_2^1\}, \\ R_{18} = R\{vx_1^2\} = V(\mathbb{G}) - \{x_3^2\}, R_{19} = R\{vx_1^3\} = V(\mathbb{G}) - \{x_4^3\}, \\ R_{20} = R\{vx_1^4\} = V(\mathbb{G}) - \{x_5^4\}, R_{21} = R\{vx_2^1\} = V(\mathbb{G}) - \{x_1^1\}, \\ R_{22} = R\{vx_2^2\} = V(\mathbb{G}) - \{x_2^2\}, R_{23} = R\{vx_3^3\} = V(\mathbb{G}) - \{x_1^3\}, \\ R_{24} = R\{vx_4^4\} = V(\mathbb{G}) - \{x_1^4\},$$

Among all the local resolving neighbourhoods,  $R_1$  is having the minimum cardinality of 2. We define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$  thereby assigning  $\frac{1}{2}$  to all the vertices in the set  $R_1$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^2 \frac{1}{2} = \frac{2}{2} = 1.$$

Notice that equality holds as there is only one and unique minimum local resolving neighbourhood.

(iii) When  $m \geq 3, 1 \leq p \leq m$  and  $q_p = n + 2(p - 1)$ .

The required local resolving neighbourhoods are for  $1 \leq j \leq q_p - 1, x_j^p x_{j+1}^p$  and  $vx_j^p$ . Among all the resolving neighbourhoods  $x_{r_\alpha}^\alpha x_{s_\alpha}^\alpha$ , where  $\alpha = \min\{q_p | 1 \leq p \leq m\}$  and  $\alpha \equiv 1 \pmod{2}$  is having the least cardinality. This implies that the minimum local resolving neighbourhood will be one and unique with  $R_1 = \alpha - 1$ . As we have seen in lemma 4.3.1 that  $\bigcup_{u=1}^m R_u = \alpha - 1$  and  $|\bigcup_{u=1}^m R_u| = \alpha - 1$ . Suppose

$\delta = |\bigcup_{u=1}^m R_u| = |R_u| = \omega$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

Thus assigning the label  $\frac{1}{\omega}$  to all the vertices of  $\alpha$  and by lemma 5.1.3, summing them all, we get

$$\dim_{lfrac}(V(\mathbb{G})) = \sum_{u=1}^{\omega} \frac{1}{\omega} = \frac{\alpha-1}{\alpha-1} = 1$$

## 5.2. Local Fractional Metric Dimension of $\mathbb{GPW}(m; q_1, q_2, \dots, q_m)$

**Theorem 5.3.1** Suppose  $\mathbb{G} \cong \mathbb{GPW}(m; q_1, q_2, \dots, q_m)$ , for  $m \geq 3$ ,  $n \geq 6$ ,  $1 \leq p \leq m$  and  $q_p = n$ , then  $\dim_{lfrac}(\mathbb{G}) = 2m$ .

**Proof:**

**Case I:** When  $m = 3$ ,  $1 \leq p \leq 3$  and  $q_p = 6$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                           |
|-----------------------------|------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1\}$          |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2\}$          |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3\}$          |
| $R_4 = R\{x_5^1 x_6^1\}$    | $\{x_4^1, x_5^1, x_6^1\}$          |
| $R_5 = R\{x_5^2 x_6^2\}$    | $\{x_4^2, x_5^2, x_6^2\}$          |
| $R_6 = R\{x_5^3 x_6^3\}$    | $\{x_4^3, x_5^3, x_6^3\}$          |
| $R_7 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$   |
| $R_8 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$   |
| $R_9 = R\{x_2^3 x_3^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3\}$   |
| $R_{10} = R\{x_3^1 x_4^1\}$ | $\{x_2^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{11} = R\{x_3^2 x_4^2\}$ | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{12} = R\{x_3^3 x_4^3\}$ | $\{x_2^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{13} = R\{x_4^1 x_5^1\}$ | $\{x_3^1, x_4^1, x_5^1, x_6^1\}$   |
| $R_{14} = R\{x_4^2 x_5^2\}$ | $\{x_3^2, x_4^2, x_5^2, x_6^2\}$   |
| $R_{15} = R\{x_4^3 x_5^3\}$ | $\{x_3^3, x_4^3, x_5^3, x_6^3\}$   |
| $R_{16} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$ |
| $R_{17} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$ |
| $R_{18} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$ |
| $R_{19} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$ |
| $R_{20} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$ |
| $R_{21} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$ |
| $R_{22} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_3^1, x_5^1\}$ |
| $R_{23} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$ |
| $R_{24} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$ |
| $R_{25} = R\{vx_5^1\}$      | $V(\mathbb{G}) - \{x_4^1, x_6^1\}$ |
| $R_{26} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_4^2, x_6^2\}$ |
| $R_{27} = R\{vx_5^3\}$      | $V(\mathbb{G}) - \{x_4^3, x_6^3\}$ |
| $R_{28} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_2^1\}$        |
| $R_{29} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_2^2\}$        |
| $R_{30} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_2^3\}$        |
| $R_{31} = R\{vx_6^1\}$      | $V(\mathbb{G}) - \{x_5^1\}$        |
| $R_{32} = R\{vx_6^2\}$      | $V(\mathbb{G}) - \{x_5^2\}$        |
| $R_{33} = R\{vx_6^3\}$      | $V(\mathbb{G}) - \{x_5^3\}$        |

From the above resolving neighbourhood sets, the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 6$  that is 3, whereas the rest of the sets from  $7 \leq l \leq 33$  are the sets having the higher cardinalities.

Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in the set

$\bigcup_{u=1}^3 R_u = \bigcup_{u=1}^3 \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u, x_6^u\}$  thus by lemma 5.1.4, summing them all, we get

$$\dim_{frac}(\mathbb{G}) = \sum_{u=1}^3 \frac{1}{3} = \frac{18}{3} = 6.$$

**Case II:** When  $m = 4$ ,  $1 \leq p \leq 4$  and  $q_p = 7$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                           |
|-----------------------------|------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1\}$          |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2\}$          |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3\}$          |
| $R_3 = R\{x_1^4 x_2^4\}$    | $\{x_1^4, x_2^4, x_3^4\}$          |
| $R_4 = R\{x_5^1 x_6^1\}$    | $\{x_5^1, x_6^1, x_7^1\}$          |
| $R_5 = R\{x_5^2 x_6^2\}$    | $\{x_5^2, x_6^2, x_7^2\}$          |
| $R_6 = R\{x_5^3 x_6^3\}$    | $\{x_5^3, x_6^3, x_7^3\}$          |
| $R_7 = R\{x_5^4 x_6^4\}$    | $\{x_5^4, x_6^4, x_7^4\}$          |
| $R_8 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$   |
| $R_9 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$   |
| $R_{10} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3\}$   |
| $R_{11} = R\{x_2^4 x_3^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4\}$   |
| $R_{12} = R\{x_3^1 x_4^1\}$ | $\{x_2^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{13} = R\{x_3^2 x_4^2\}$ | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{14} = R\{x_3^3 x_4^3\}$ | $\{x_2^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{15} = R\{x_3^4 x_4^4\}$ | $\{x_2^4, x_3^4, x_4^4, x_5^4\}$   |
| $R_{16} = R\{x_4^1 x_5^1\}$ | $\{x_3^1, x_4^1, x_5^1, x_6^1\}$   |
| $R_{17} = R\{x_4^2 x_5^2\}$ | $\{x_3^2, x_4^2, x_5^2, x_6^2\}$   |
| $R_{18} = R\{x_4^3 x_5^3\}$ | $\{x_3^3, x_4^3, x_5^3, x_6^3\}$   |
| $R_{19} = R\{x_4^4 x_5^4\}$ | $\{x_3^4, x_4^4, x_5^4, x_6^4\}$   |
| $R_{20} = R\{x_5^1 x_6^1\}$ | $\{x_4^1, x_5^1, x_6^1, x_7^1\}$   |
| $R_{21} = R\{x_5^2 x_6^2\}$ | $\{x_4^2, x_5^2, x_6^2, x_7^2\}$   |
| $R_{22} = R\{x_5^3 x_6^3\}$ | $\{x_4^3, x_5^3, x_6^3, x_7^3\}$   |
| $R_{23} = R\{x_5^4 x_6^4\}$ | $\{x_4^4, x_5^4, x_6^4, x_7^4\}$   |
| $R_{24} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$ |
| $R_{25} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$ |
| $R_{26} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$ |
| $R_{27} = R\{vx_2^4\}$      | $V(\mathbb{G}) - \{x_1^4, x_3^4\}$ |
| $R_{28} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$ |
| $R_{29} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$ |
| $R_{30} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$ |
| $R_{31} = R\{vx_3^4\}$      | $V(\mathbb{G}) - \{x_2^4, x_4^4\}$ |
| $R_{32} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_3^1, x_5^1\}$ |
| $R_{33} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$ |
| $R_{34} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$ |
| $R_{35} = R\{vx_4^4\}$      | $V(\mathbb{G}) - \{x_3^4, x_5^4\}$ |
| $R_{36} = R\{vx_5^1\}$      | $V(\mathbb{G}) - \{x_4^1, x_6^1\}$ |
| $R_{37} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_4^2, x_6^2\}$ |

$$\begin{array}{ll}
R_{38} = R\{vx_5^3\} & V(\mathbb{G}) - \{x_4^3, x_6^3\} \\
R_{39} = R\{vx_5^4\} & V(\mathbb{G}) - \{x_4^4, x_6^4\} \\
R_{40} = R\{vx_6^1\} & V(\mathbb{G}) - \{x_5^1, x_7^1\} \\
R_{41} = R\{vx_6^2\} & V(\mathbb{G}) - \{x_5^2, x_7^2\} \\
R_{42} = R\{vx_6^3\} & V(\mathbb{G}) - \{x_5^3, x_7^3\} \\
R_{43} = R\{vx_6^4\} & V(\mathbb{G}) - \{x_5^4, x_7^4\} \\
R_{44} = R\{vx_1^1\} & V(\mathbb{G}) - \{x_2^1\} \\
R_{45} = R\{vx_1^2\} & V(\mathbb{G}) - \{x_2^2\} \\
R_{46} = R\{vx_1^3\} & V(\mathbb{G}) - \{x_2^3\} \\
R_{47} = R\{vx_1^4\} & V(\mathbb{G}) - \{x_2^4\} \\
R_{48} = R\{vx_7^1\} & V(\mathbb{G}) - \{x_6^1\} \\
R_{49} = R\{vx_7^2\} & V(\mathbb{G}) - \{x_6^2\} \\
R_{50} = R\{vx_7^3\} & V(\mathbb{G}) - \{x_6^3\} \\
R_{51} = R\{vx_7^4\} & V(\mathbb{G}) - \{x_6^4\}
\end{array}$$

As it is evident from the above resolving neighbourhood sets, that the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 6$  that is 6, whereas the rest of the sets from  $7 \leq l \leq 51$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{3}$  to the vertices in

the set  $\bigcup_{u=1}^4 R_u = \bigcup_{u=1}^4 \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u, x_6^u\}$ , hence by lemma 5.1.4, summing them all, we get

$$\dim_{frac}(\mathbb{G}) = \sum_{u=1}^4 \frac{1}{6} = \frac{24}{3} = 8.$$

**Case III:** When  $m \geq 3$ ,  $1 \leq p \leq m$  and  $q_p = n$ .

The required local resolving neighbourhoods are  $x_j^p x_{j+1}^p$  and  $vx_j^p$ . Among all the resolving neighbourhoods  $x_1^p x_2^p$  and  $x_{p-1}^p x_p^p$  having the minimum cardinality of 3. As we have seen in lemma 4.3.2 that

$\bigcup_{u=1}^p R_u = \bigcup_{u=1}^p \{x_1^u, x_2^u, x_3^u, x_{p-2}^u, x_{p-1}^u, x_p^u\}$  and  $|\bigcup_{u=1}^p R_u| = 6m$ . Suppose  $\delta = |\bigcup_{u=1}^m R_u|$  and  $\omega = |R_u| = \omega$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

Thus assigning the label  $\frac{1}{\omega}$  to all the vertices of in the set  $\bigcup_{u=1}^p R_u$  and by lemma 5.1.4, evaluating their sum, we get

$$\dim_{frac}(V(\mathbb{G})) = \sum_{u=1}^{\delta} \frac{1}{\omega} = \frac{6m}{3} = 2m.$$

**Theorem 5.3.2** Suppose  $\mathbb{G} \cong \text{GPW}(m; q_1, q_2, \dots, q_m)$ , for  $m \geq 3$ ,  $n \geq 6$ ,  $1 \leq p \leq m$  and  $q_p = n + p - 1$ , then

$$\dim_{frac}(\mathbb{G}) = 2m.$$

**Proof:**

**Case I:** When  $m = 3$ ,  $q_1 = 4$ ,  $q_2 = 5$  and  $q_3 = 6$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods     | Elements                           |
|------------------------------|------------------------------------|
| $R_1 = R\{x_1^1, x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1\}$          |
| $R_2 = R\{x_1^2, x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2\}$          |
| $R_3 = R\{x_1^3, x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3\}$          |
| $R_4 = R\{x_4^1, x_5^1\}$    | $\{x_4^1, x_5^1, x_6^1\}$          |
| $R_5 = R\{x_4^2, x_5^2\}$    | $\{x_4^2, x_5^2, x_6^2\}$          |
| $R_6 = R\{x_4^3, x_5^3\}$    | $\{x_4^3, x_5^3, x_6^3\}$          |
| $R_7 = R\{x_2^1, x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$   |
| $R_8 = R\{x_2^2, x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$   |
| $R_9 = R\{x_2^3, x_3^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3\}$   |
| $R_{10} = R\{x_3^1, x_4^1\}$ | $\{x_2^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{11} = R\{x_3^2, x_4^2\}$ | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{12} = R\{x_3^3, x_4^3\}$ | $\{x_2^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{13} = R\{x_4^1, x_5^1\}$ | $\{x_3^1, x_4^1, x_5^1, x_6^1\}$   |
| $R_{14} = R\{x_4^2, x_5^2\}$ | $\{x_3^2, x_4^2, x_5^2, x_6^2\}$   |
| $R_{15} = R\{x_4^3, x_5^3\}$ | $\{x_3^3, x_4^3, x_5^3, x_6^3\}$   |
| $R_{16} = R\{x_5^2, x_6^2\}$ | $\{x_4^2, x_5^2, x_6^2, x_7^2\}$   |
| $R_{17} = R\{x_5^3, x_6^3\}$ | $\{x_4^3, x_5^3, x_6^3, x_7^3\}$   |
| $R_{18} = R\{x_6^3, x_7^3\}$ | $\{x_5^3, x_6^3, x_7^3, x_8^3\}$   |
| $R_{19} = R\{vx_2^1\}$       | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$ |
| $R_{20} = R\{vx_2^2\}$       | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$ |
| $R_{21} = R\{vx_2^3\}$       | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$ |
| $R_{22} = R\{vx_3^1\}$       | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$ |
| $R_{23} = R\{vx_3^2\}$       | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$ |
| $R_{24} = R\{vx_3^3\}$       | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$ |
| $R_{25} = R\{vx_4^1\}$       | $V(\mathbb{G}) - \{x_3^1, x_5^1\}$ |
| $R_{26} = R\{vx_4^2\}$       | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$ |
| $R_{27} = R\{vx_4^3\}$       | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$ |
| $R_{28} = R\{vx_5^1\}$       | $V(\mathbb{G}) - \{x_4^1, x_6^1\}$ |
| $R_{29} = R\{vx_5^2\}$       | $V(\mathbb{G}) - \{x_4^2, x_6^2\}$ |
| $R_{30} = R\{vx_5^3\}$       | $V(\mathbb{G}) - \{x_4^3, x_6^3\}$ |
| $R_{31} = R\{vx_6^2\}$       | $V(\mathbb{G}) - \{x_5^2, x_7^2\}$ |
| $R_{32} = R\{vx_6^3\}$       | $V(\mathbb{G}) - \{x_5^3, x_7^3\}$ |
| $R_{33} = R\{vx_7^3\}$       | $V(\mathbb{G}) - \{x_6^3, x_8^3\}$ |
| $R_{34} = R\{vx_1^1\}$       | $V(\mathbb{G}) - \{x_2^1\}$        |
| $R_{35} = R\{vx_1^2\}$       | $V(\mathbb{G}) - \{x_2^2\}$        |
| $R_{36} = R\{vx_1^3\}$       | $V(\mathbb{G}) - \{x_2^3\}$        |
| $R_{37} = R\{vx_6^1\}$       | $V(\mathbb{G}) - \{x_5^1\}$        |
| $R_{38} = R\{vx_7^2\}$       | $V(\mathbb{G}) - \{x_6^2\}$        |
| $R_{39} = R\{vx_8^3\}$       | $V(\mathbb{G}) - \{x_7^3\}$        |

As we can see from the above that the sets with the minimum cardinality are of 6 are  $R_l$  for  $1 \leq l \leq 6$ , whereas the rest of the sets from  $7 \leq l \leq 39$  are the sets with the higher cardinalities. Therefore, we define a

mapping  $\psi: V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in the set  $\bigcup_{u=1}^3 R_u = \bigcup_{u=1}^3 \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u, x_6^u\}$ , therefore by lemma 5.1.4, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^3 \frac{1}{6} = \frac{18}{6} = 3.$$

**Case II:** When  $m = 4$ ,  $q_1 = 6$ ,  $q_2 = 7$ ,  $q_3 = 8$  and  $q_4 = 9$ .

The required local resolving neighbourhoods by lemma 5.1.4, are given by:

| Resolving Neighbourhoods    | Elements                           |
|-----------------------------|------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1\}$          |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2\}$          |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3\}$          |
| $R_4 = R\{x_1^4 x_2^4\}$    | $\{x_1^4, x_2^4, x_3^4\}$          |
| $R_5 = R\{x_5^1 x_6^1\}$    | $\{x_4^1, x_5^1, x_6^1\}$          |
| $R_6 = R\{x_5^2 x_6^2\}$    | $\{x_5^2, x_6^2, x_7^2\}$          |
| $R_7 = R\{x_5^3 x_6^3\}$    | $\{x_6^3, x_7^3, x_8^3\}$          |
| $R_8 = R\{x_5^4 x_6^4\}$    | $\{x_7^4, x_8^4, x_9^4\}$          |
| $R_9 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$   |
| $R_{10} = R\{x_2^2 x_3^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$   |
| $R_{11} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3\}$   |
| $R_{12} = R\{x_2^4 x_3^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4\}$   |
| $R_{13} = R\{x_3^1 x_4^1\}$ | $\{x_2^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{14} = R\{x_3^2 x_4^2\}$ | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{15} = R\{x_3^3 x_4^3\}$ | $\{x_2^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{16} = R\{x_3^4 x_4^4\}$ | $\{x_2^4, x_3^4, x_4^4, x_5^4\}$   |
| $R_{17} = R\{x_4^1 x_5^1\}$ | $\{x_3^1, x_4^1, x_5^1, x_6^1\}$   |
| $R_{18} = R\{x_4^2 x_5^2\}$ | $\{x_3^2, x_4^2, x_5^2, x_6^2\}$   |
| $R_{19} = R\{x_4^3 x_5^3\}$ | $\{x_3^3, x_4^3, x_5^3, x_6^3\}$   |
| $R_{20} = R\{x_4^4 x_5^4\}$ | $\{x_3^4, x_4^4, x_5^4, x_6^4\}$   |
| $R_{21} = R\{x_5^2 x_6^2\}$ | $\{x_4^2, x_5^2, x_6^2, x_7^2\}$   |
| $R_{22} = R\{x_5^3 x_6^3\}$ | $\{x_4^3, x_5^3, x_6^3, x_7^3\}$   |
| $R_{23} = R\{x_5^4 x_6^4\}$ | $\{x_4^4, x_5^4, x_6^4, x_7^4\}$   |
| $R_{24} = R\{x_6^3 x_7^3\}$ | $\{x_5^3, x_6^3, x_7^3, x_8^3\}$   |
| $R_{25} = R\{x_6^4 x_7^4\}$ | $\{x_5^4, x_6^4, x_7^4, x_8^4\}$   |
| $R_{26} = R\{x_7^3 x_8^3\}$ | $\{x_6^3, x_7^3, x_8^3, x_9^3\}$   |
| $R_{27} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$ |
| $R_{28} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$ |
| $R_{29} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$ |
| $R_{30} = R\{vx_2^4\}$      | $V(\mathbb{G}) - \{x_1^4, x_3^4\}$ |
| $R_{31} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$ |
| $R_{32} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$ |
| $R_{33} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$ |
| $R_{34} = R\{vx_3^4\}$      | $V(\mathbb{G}) - \{x_2^4, x_4^4\}$ |
| $R_{35} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_3^1, x_5^1\}$ |
| $R_{36} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$ |
| $R_{37} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$ |
| $R_{38} = R\{vx_4^4\}$      | $V(\mathbb{G}) - \{x_3^4, x_5^4\}$ |
| $R_{39} = R\{vx_5^1\}$      | $V(\mathbb{G}) - \{x_4^1, x_6^1\}$ |
| $R_{40} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_4^2, x_6^2\}$ |
| $R_{41} = R\{vx_5^3\}$      | $V(\mathbb{G}) - \{x_4^3, x_6^3\}$ |
| $R_{42} = R\{vx_5^4\}$      | $V(\mathbb{G}) - \{x_4^4, x_6^4\}$ |
| $R_{43} = R\{vx_6^2\}$      | $V(\mathbb{G}) - \{x_5^2, x_7^2\}$ |

$$\begin{array}{ll}
 R_{44} = R\{vx_6^3\} & V(\mathbb{G}) - \{x_5^3, x_7^3\} \\
 R_{45} = R\{vx_6^4\} & V(\mathbb{G}) - \{x_5^4, x_7^4\} \\
 R_{46} = R\{vx_7^3\} & V(\mathbb{G}) - \{x_6^3, x_8^3\} \\
 R_{47} = R\{vx_7^4\} & V(\mathbb{G}) - \{x_6^4, x_8^4\} \\
 R_{48} = R\{vx_8^4\} & V(\mathbb{G}) - \{x_7^4, x_9^4\} \\
 R_{49} = R\{vx_1^1\} & V(\mathbb{G}) - \{x_2^1\} \\
 R_{50} = R\{vx_1^2\} & V(\mathbb{G}) - \{x_2^2\} \\
 R_{51} = R\{vx_1^3\} & V(\mathbb{G}) - \{x_2^3\} \\
 R_{52} = R\{vx_1^4\} & V(\mathbb{G}) - \{x_2^4\} \\
 R_{53} = R\{vx_6^1\} & V(\mathbb{G}) - \{x_5^1\} \\
 R_{54} = R\{vx_7^2\} & V(\mathbb{G}) - \{x_6^2\} \\
 R_{55} = R\{vx_8^3\} & V(\mathbb{G}) - \{x_7^3\} \\
 R_{56} = R\{vx_9^4\} & V(\mathbb{G}) - \{x_8^3\}
 \end{array}$$

As it is evident from the above that the resolving neighbourhoods with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 6$  that is 3, whereas the rest of the sets from  $7 \leq l \leq 56$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in

the set  $\bigcup_{u=1}^{24} R_u = \bigcup_{u=1}^{24} \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u, x_6^u\}$ , thereby by lemma 5.1.4, summing them all, we get

$$dim_{frac}(\mathbb{G}) = \sum_{u=1}^6 \frac{1}{3} = \frac{24}{3} = 8.$$

**Case III:** When  $m \geq 3$ ,  $1 \leq p \leq m$  and  $q_p = n + p - 1$ .

The possible local resolving neighbourhoods are  $x_j^p x_{j+1}^p$  and  $vx_j^p$ . Among all the resolving neighbourhoods  $x_1^p x_2^p$  and  $x_{p-1}^p x_p^p$  having the lowest cardinality of 3. As we have seen in lemma 4.3.2 that

$\bigcup_{u=1}^p R_u = \bigcup_{u=1}^p \{x_1^u, x_2^u, x_3^u, x_{p-2}^u, x_{p-1}^u, x_p^u\}$  and  $|\bigcup_{u=1}^p R_u| = 6m$ . Suppose  $\delta = |\bigcup_{u=1}^m R_u|$  and  $\omega = |R_u| = \omega$  Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

Thereby assigning the label  $\frac{1}{\omega}$  to all the vertices of in the set  $\bigcup_{u=1}^p R_u$  and by lemma 5.1.4, manipulating their sum, we get

$$dim_{frac}(V(\mathbb{G})) = \sum_{u=1}^{\delta} \frac{1}{\omega} = \frac{6m}{3} = 2m$$

### 5.3. Local Fractional Metric Dimension of $\mathbb{G}^{WW}(m; q_1, q_2, \dots, q_m)$

**Theorem 5.4.1** Suppose  $\mathbb{G} \cong \mathbb{G}^{WW}(m; q_1, q_2, \dots, q_m)$ , for  $m \geq 3$ ,  $n \geq 3$ ,  $1 \leq p \leq m$  and  $q_p = n$ , then

$$dim_{frac}(\mathbb{G}) = \begin{cases} \frac{mn}{n-1} & \text{for } q_p \equiv 1 \pmod{2} \\ m & \text{for } q_p \equiv 0 \pmod{2} \end{cases}$$

**Proof:**

**Case I:** For  $n \equiv 1 \pmod{2}$

(i): When  $m = 3$ ,  $1 \leq p \leq 3$  and  $q_p = 5$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                           |
|-----------------------------|------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_5^1\}$   |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_5^2\}$   |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_5^3\}$   |
| $R_4 = R\{x_1^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$   |
| $R_5 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$   |
| $R_6 = R\{x_2^3 x_3^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3\}$   |
| $R_7 = R\{x_3^1 x_4^1\}$    | $\{x_2^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_8 = R\{x_3^2 x_4^2\}$    | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_9 = R\{x_3^3 x_4^3\}$    | $\{x_2^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{10} = R\{x_4^1 x_5^1\}$ | $\{x_1^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{11} = R\{x_4^2 x_5^2\}$ | $\{x_1^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{12} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{13} = R\{x_1^1 x_5^1\}$ | $\{x_1^1, x_2^1, x_4^1, x_5^1\}$   |
| $R_{14} = R\{x_1^2 x_5^2\}$ | $\{x_1^2, x_2^2, x_4^2, x_5^2\}$   |
| $R_{15} = R\{x_1^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_4^3, x_5^3\}$   |
| $R_{16} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_5^1, x_2^1\}$ |
| $R_{17} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_5^2, x_2^2\}$ |
| $R_{18} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_5^3, x_2^3\}$ |
| $R_{19} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$ |
| $R_{20} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$ |
| $R_{21} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$ |
| $R_{22} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$ |
| $R_{23} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$ |
| $R_{24} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$ |
| $R_{25} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_3^1, x_5^1\}$ |
| $R_{26} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$ |
| $R_{27} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$ |
| $R_{28} = R\{vx_5^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_4^1\}$ |
| $R_{29} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_4^2\}$ |
| $R_{30} = R\{vx_5^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_4^3\}$ |

From the above resolving neighbourhood sets, the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 15$  that is 4, whereas the rest of the sets from  $7 \leq l \leq 33$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{4}$  to the vertices in the set

$\bigcup_{u=1}^3 R_u = \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u\}$  and by lemma 5.1.5, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^3 \frac{1}{4} = \frac{15}{4}.$$

(ii) When  $m = 4$ ,  $1 \leq p \leq 4$  and  $q_p = 5$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                           |
|-----------------------------|------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_5^1\}$   |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_5^2\}$   |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_5^3\}$   |
| $R_4 = R\{x_1^4 x_2^4\}$    | $\{x_1^4, x_2^4, x_3^4, x_5^4\}$   |
| $R_5 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$   |
| $R_6 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$   |
| $R_7 = R\{x_2^3 x_3^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3\}$   |
| $R_8 = R\{x_2^4 x_3^4\}$    | $\{x_1^4, x_2^4, x_3^4, x_4^4\}$   |
| $R_9 = R\{x_3^1 x_4^1\}$    | $\{x_2^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{10} = R\{x_3^2 x_4^2\}$ | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{11} = R\{x_3^3 x_4^3\}$ | $\{x_2^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{12} = R\{x_3^4 x_4^4\}$ | $\{x_2^4, x_3^4, x_4^4, x_5^4\}$   |
| $R_{13} = R\{x_4^1 x_5^1\}$ | $\{x_1^1, x_3^1, x_4^1, x_5^1\}$   |
| $R_{14} = R\{x_4^2 x_5^2\}$ | $\{x_1^2, x_3^2, x_4^2, x_5^2\}$   |
| $R_{15} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_3^3, x_4^3, x_5^3\}$   |
| $R_{16} = R\{x_4^4 x_5^4\}$ | $\{x_1^4, x_3^4, x_4^4, x_5^4\}$   |
| $R_{17} = R\{x_1^1 x_5^1\}$ | $\{x_1^1, x_2^1, x_4^1, x_5^1\}$   |
| $R_{18} = R\{x_1^2 x_5^2\}$ | $\{x_1^2, x_2^2, x_4^2, x_5^2\}$   |
| $R_{19} = R\{x_1^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_4^3, x_5^3\}$   |
| $R_{20} = R\{x_1^4 x_5^4\}$ | $\{x_1^4, x_2^4, x_4^4, x_5^4\}$   |
| $R_{21} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_5^1, x_2^1\}$ |
| $R_{22} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_5^2, x_2^2\}$ |
| $R_{23} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_5^3, x_2^3\}$ |
| $R_{24} = R\{vx_1^4\}$      | $V(\mathbb{G}) - \{x_5^4, x_2^4\}$ |
| $R_{25} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$ |
| $R_{26} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$ |
| $R_{27} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$ |
| $R_{28} = R\{vx_2^4\}$      | $V(\mathbb{G}) - \{x_1^4, x_3^4\}$ |
| $R_{29} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$ |
| $R_{30} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$ |
| $R_{31} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$ |
| $R_{32} = R\{vx_3^4\}$      | $V(\mathbb{G}) - \{x_2^4, x_4^4\}$ |
| $R_{33} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_3^1, x_5^1\}$ |
| $R_{34} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$ |
| $R_{35} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$ |
| $R_{36} = R\{vx_4^4\}$      | $V(\mathbb{G}) - \{x_3^4, x_5^4\}$ |
| $R_{37} = R\{vx_5^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_4^1\}$ |
| $R_{38} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_4^2\}$ |
| $R_{39} = R\{vx_5^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_4^3\}$ |
| $R_{40} = R\{vx_5^4\}$      | $V(\mathbb{G}) - \{x_1^4, x_4^4\}$ |

As it is evident from the above resolving neighbourhood sets, that the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 20$  that is 4, whereas the rest of the sets from  $21 \leq l \leq 40$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{4}$  to the vertices in

the set  $\bigcup_{u=1}^4 R_u = \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u\}$  lemma 5.1.5, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^4 \frac{1}{4} = \frac{20}{4} = 5.$$

(iii) When  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $n \geq 3$ ,  $n \equiv 0 \pmod{2}$  and  $q_p = n$ .

The local resolving neighbourhoods of  $\mathbb{G}$  are  $x_j^p x_{j+1}^p$ , and  $vx_j$ . Out of these the set with minimum cardinality is  $x_j^p x_{j+1}^p$  with  $|R_u| = |R\{x_j^p x_{j+1}^p\}| = n - 1$  and  $\bigcup_{u=1}^p R_u = \bigcup_{u=1}^p \{x_k^u | 1 \leq k \leq p\}$ . Therefore, let

$\delta = |\bigcup_{u=1}^p R_u| = mn$  and  $\omega = |R_u| = n - 1$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

. Assigning all the vertices of  $\mathbb{G}$  except  $\{v\}$  the label  $\frac{1}{\omega}$  and by lemma 5.1.5, calculating their sum, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^p \frac{1}{\omega} = \frac{mn}{n-1}.$$

(i): When  $m = 3$ ,  $1 \leq p \leq 3$  and  $q_p = 6$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                                       |
|-----------------------------|------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_4 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_5 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_6 = R\{x_2^3 x_3^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_7 = R\{x_3^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_8 = R\{x_3^2 x_4^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_9 = R\{x_3^3 x_4^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{10} = R\{x_4^1 x_5^1\}$ | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_{11} = R\{x_4^2 x_5^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_{12} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{13} = R\{x_5^1 x_6^1\}$ | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_{14} = R\{x_5^2 x_6^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_{15} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{16} = R\{x_1^1 x_6^1\}$ | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_{17} = R\{x_1^2 x_6^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_{18} = R\{x_1^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{19} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_6^1, x_2^1\}$             |
| $R_{20} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_6^2, x_2^2\}$             |
| $R_{21} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_6^3, x_2^3\}$             |
| $R_{22} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$             |
| $R_{23} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$             |

$$\begin{aligned}
 R_{24} &= R\{vx_2^3\} & V(\mathbb{G}) - \{x_1^3, x_3^3\} \\
 R_{25} &= R\{vx_3^1\} & V(\mathbb{G}) - \{x_2^1, x_4^1\} \\
 R_{26} &= R\{vx_3^2\} & V(\mathbb{G}) - \{x_2^2, x_4^2\} \\
 R_{27} &= R\{vx_3^3\} & V(\mathbb{G}) - \{x_2^3, x_4^3\} \\
 R_{28} &= R\{vx_4^1\} & V(\mathbb{G}) - \{x_3^1, x_5^1\} \\
 R_{29} &= R\{vx_4^2\} & V(\mathbb{G}) - \{x_3^2, x_5^2\} \\
 R_{30} &= R\{vx_4^3\} & V(\mathbb{G}) - \{x_3^3, x_5^3\} \\
 R_{31} &= R\{vx_5^1\} & V(\mathbb{G}) - \{x_4^1, x_6^1\} \\
 R_{32} &= R\{vx_5^2\} & V(\mathbb{G}) - \{x_4^2, x_6^2\} \\
 R_{33} &= R\{vx_5^3\} & V(\mathbb{G}) - \{x_4^3, x_6^3\} \\
 R_{34} &= R\{vx_6^1\} & V(\mathbb{G}) - \{x_1^1, x_5^1\} \\
 R_{35} &= R\{vx_6^2\} & V(\mathbb{G}) - \{x_1^2, x_5^2\} \\
 R_{36} &= R\{vx_6^3\} & V(\mathbb{G}) - \{x_1^3, x_5^3\}
 \end{aligned}$$

From the above resolving neighbourhood sets, the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 18$  that is 6, whereas the rest of the sets from  $19 \leq l \leq 36$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{4}$  to the vertices in the set

$$\bigcup_{u=1}^3 R_u = \bigcup_{u=1}^3 \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u, x_6^u\} \text{ and by lemma 5.1.5, summing them all, we get}$$

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^3 \frac{1}{6} = \frac{36}{6} = 6.$$

(ii) When  $m = 4$ ,  $1 \leq p \leq 4$  and  $q_p = 6$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                                       |
|-----------------------------|------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_3 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_4 = R\{x_1^4 x_2^4\}$    | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\}$ |
| $R_5 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_6 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_7 = R\{x_2^3 x_3^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_8 = R\{x_2^4 x_3^4\}$    | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\}$ |
| $R_9 = R\{x_3^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_{10} = R\{x_3^2 x_4^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_{11} = R\{x_3^3 x_4^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{12} = R\{x_3^4 x_4^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\}$ |
| $R_{13} = R\{x_4^1 x_5^1\}$ | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_{14} = R\{x_4^2 x_5^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_{15} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{16} = R\{x_4^4 x_5^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\}$ |
| $R_{17} = R\{x_5^1 x_6^1\}$ | $\{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\}$ |
| $R_{18} = R\{x_5^2 x_6^2\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$ |
| $R_{19} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{20} = R\{x_5^4 x_6^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\}$ |

$$\begin{array}{ll}
R_{21} = R\{x_1^1 x_6^1\} & \{x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1\} \\
R_{22} = R\{x_1^2 x_6^2\} & \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\} \\
R_{23} = R\{x_1^3 x_6^3\} & \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\} \\
R_{24} = R\{x_1^4 x_6^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4\} \\
R_{25} = R\{vx_1^1\} & V(\mathbb{G}) - \{x_6^1, x_2^1\} \\
R_{26} = R\{vx_1^2\} & V(\mathbb{G}) - \{x_6^2, x_2^2\} \\
R_{27} = R\{vx_1^3\} & V(\mathbb{G}) - \{x_6^3, x_2^3\} \\
R_{28} = R\{vx_1^4\} & V(\mathbb{G}) - \{x_6^4, x_2^4\} \\
R_{29} = R\{vx_2^1\} & V(\mathbb{G}) - \{x_1^1, x_3^1\} \\
R_{30} = R\{vx_2^2\} & V(\mathbb{G}) - \{x_1^2, x_3^2\} \\
R_{31} = R\{vx_2^3\} & V(\mathbb{G}) - \{x_1^3, x_3^3\} \\
R_{32} = R\{vx_2^4\} & V(\mathbb{G}) - \{x_1^4, x_3^4\} \\
R_{33} = R\{vx_3^1\} & V(\mathbb{G}) - \{x_2^1, x_4^1\} \\
R_{34} = R\{vx_3^2\} & V(\mathbb{G}) - \{x_2^2, x_4^2\} \\
R_{35} = R\{vx_3^3\} & V(\mathbb{G}) - \{x_2^3, x_4^3\} \\
R_{36} = R\{vx_3^4\} & V(\mathbb{G}) - \{x_2^4, x_4^4\} \\
R_{37} = R\{vx_4^1\} & V(\mathbb{G}) - \{x_3^1, x_5^1\} \\
R_{38} = R\{vx_4^2\} & V(\mathbb{G}) - \{x_3^2, x_5^2\} \\
R_{39} = R\{vx_4^3\} & V(\mathbb{G}) - \{x_3^3, x_5^3\} \\
R_{40} = R\{vx_4^4\} & V(\mathbb{G}) - \{x_3^4, x_5^4\} \\
R_{41} = R\{vx_5^1\} & V(\mathbb{G}) - \{x_4^1, x_6^1\} \\
R_{42} = R\{vx_5^2\} & V(\mathbb{G}) - \{x_4^2, x_6^2\} \\
R_{43} = R\{vx_5^3\} & V(\mathbb{G}) - \{x_4^3, x_6^3\} \\
R_{44} = R\{vx_5^4\} & V(\mathbb{G}) - \{x_4^4, x_6^4\} \\
R_{45} = R\{vx_6^1\} & V(\mathbb{G}) - \{x_1^1, x_5^1\} \\
R_{46} = R\{vx_6^2\} & V(\mathbb{G}) - \{x_1^2, x_5^2\} \\
R_{47} = R\{vx_6^3\} & V(\mathbb{G}) - \{x_1^3, x_5^3\} \\
R_{48} = R\{vx_6^4\} & V(\mathbb{G}) - \{x_1^4, x_5^4\}
\end{array}$$

As it is evident from the above resolving neighbourhood sets, that the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 24$  that is 6, whereas the rest of the sets from  $25 \leq l \leq 48$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in

the set  $\bigcup_{u=1}^4 R_u = \bigcup_{u=1}^6 \{x_1^u, x_2^u, x_3^u, x_4^u, x_5^u, x_6^u\}$  and from lemma 5.1.5, summing them all, we get

$$\dim_{\text{frac}}(\mathbb{G}) = \sum_{u=1}^6 \frac{1}{6} = \frac{24}{6} = 4.$$

(iii) When  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $n \geq 3$ ,  $n \equiv 1 \pmod{2}$  and  $q_p = n$ .

The possible local resolving neighbourhoods of  $\mathbb{G}$  are  $x_j^p x_{j+1}^p$ , and  $vx_j$ . Among these set  $R\{x_j^p x_{j+1}^p\}$  is having the minimum cardinality, that is  $|R_u| = |R\{x_j^p x_{j+1}^p\}| = n$  and  $\bigcup_{u=1}^p R_u = \bigcup_{u=1}^p \{x_k^u | 1 \leq k \leq p\}$ . Therefore, let  $\delta = |\bigcup_{u=1}^p R_u| = mn$  and  $\omega = |R_u| = n$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

. Assigning all the vertices of  $\mathbb{G}$  except  $\{v\}$  the label  $\frac{1}{\omega}$  and from lemma 5.1.5, calculating their sum, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^p \frac{1}{\omega} = \frac{mn}{n} = m.$$

**Theorem 5.4.2** Suppose  $\mathbb{G} \cong \mathbb{G}^{WWW}(m; q_1, q_2, \dots, q_m)$ , for  $m \geq 3$ ,  $n \geq 4$ ,  $1 \leq p \leq m$  and  $q_p = n + p - 1$ , then

$$\dim_{lfrac}(\mathbb{G}) = \frac{2n+1}{n}.$$

**Proof:**

**Case I:** When  $m = 3$ ,  $q_1 = 4$ ,  $q_2 = 5$  and  $q_3 = 6$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                                       |
|-----------------------------|------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$               |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_5^2\}$               |
| $R_3 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$               |
| $R_4 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$               |
| $R_5 = R\{x_3^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$               |
| $R_6 = R\{x_3^2 x_4^2\}$    | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$               |
| $R_7 = R\{x_1^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$               |
| $R_8 = R\{x_4^2 x_5^2\}$    | $\{x_1^2, x_3^2, x_4^2, x_5^2\}$               |
| $R_9 = R\{x_1^2 x_5^2\}$    | $\{x_1^2, x_2^2, x_4^2, x_5^2\}$               |
| $R_{10} = R\{x_1^3 x_2^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{11} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{12} = R\{x_3^3 x_4^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{13} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{14} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{15} = R\{x_1^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{16} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$             |
| $R_{17} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_5^2\}$             |
| $R_{18} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_6^3\}$             |
| $R_{19} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$             |
| $R_{20} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$             |
| $R_{21} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$             |
| $R_{22} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$             |
| $R_{23} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$             |
| $R_{24} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$             |
| $R_{25} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$             |
| $R_{26} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$             |
| $R_{27} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$             |
| $R_{28} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_4^2\}$             |
| $R_{29} = R\{vx_5^3\}$      | $V(\mathbb{G}) - \{x_4^3, x_6^3\}$             |
| $R_{30} = R\{vx_6^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_5^3\}$             |

As we can see from the above that the sets with the minimum cardinality are of 4 are  $R_l$  for  $1 \leq l \leq 9$ , whereas the rest of the sets from  $7 \leq l \leq 30$  are the sets with the higher cardinalities. Therefore,

we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in the set  $\bigcup_{u=1}^9 R_u = \{x_1^1, x_2^1, x_3^1, x_4^1, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2\}$  and summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^2 \frac{1}{4} = \frac{9}{4}.$$

**Case II:** When  $m = 4$ ,  $q_1 = 4, q_2 = 5, q_3 = 6$  and  $q_4 = 7$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                                              |
|-----------------------------|-------------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                      |
| $R_2 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_5^2\}$                      |
| $R_3 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                      |
| $R_4 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$                      |
| $R_5 = R\{x_3^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                      |
| $R_6 = R\{x_3^2 x_4^2\}$    | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$                      |
| $R_7 = R\{x_1^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                      |
| $R_8 = R\{x_4^2 x_5^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2\}$               |
| $R_9 = R\{x_1^2 x_5^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2\}$               |
| $R_{10} = R\{x_1^3 x_2^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$        |
| $R_{11} = R\{x_1^4 x_2^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{12} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$        |
| $R_{13} = R\{x_2^4 x_3^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{14} = R\{x_3^3 x_4^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$        |
| $R_{15} = R\{x_3^4 x_4^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{16} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$        |
| $R_{17} = R\{x_4^4 x_5^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{18} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$        |
| $R_{19} = R\{x_5^4 x_6^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{20} = R\{x_1^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$        |
| $R_{21} = R\{x_6^4 x_7^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{22} = R\{x_1^4 x_7^4\}$ | $\{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4\}$ |
| $R_{23} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$                    |
| $R_{24} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_5^2\}$                    |
| $R_{25} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_6^3\}$                    |
| $R_{26} = R\{vx_1^4\}$      | $V(\mathbb{G}) - \{x_2^4, x_7^4\}$                    |
| $R_{27} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$                    |
| $R_{28} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_3^2\}$                    |
| $R_{29} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$                    |
| $R_{30} = R\{vx_2^4\}$      | $V(\mathbb{G}) - \{x_1^4, x_3^4\}$                    |
| $R_{31} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$                    |
| $R_{32} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$                    |
| $R_{33} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$                    |
| $R_{34} = R\{vx_3^4\}$      | $V(\mathbb{G}) - \{x_2^4, x_4^4\}$                    |
| $R_{35} = R\{vx_4^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$                    |
| $R_{36} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_5^2\}$                    |
| $R_{37} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$                    |
| $R_{38} = R\{vx_4^4\}$      | $V(\mathbb{G}) - \{x_3^4, x_5^4\}$                    |

$$\begin{array}{ll}
 R_{39} = R\{vx_5^2\} & V(\mathbb{G}) - \{x_1^2, x_4^2\} \\
 R_{40} = R\{vx_5^3\} & V(\mathbb{G}) - \{x_4^3, x_6^3\} \\
 R_{41} = R\{vx_5^4\} & V(\mathbb{G}) - \{x_4^4, x_6^4\} \\
 R_{42} = R\{vx_6^3\} & V(\mathbb{G}) - \{x_1^3, x_5^3\} \\
 R_{43} = R\{vx_6^4\} & V(\mathbb{G}) - \{x_5^4, x_7^4\} \\
 R_{44} = R\{vx_7^4\} & V(\mathbb{G}) - \{x_1^4, x_6^3\}
 \end{array}$$

As it is evident from the above that the resolving neighbourhoods with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 10$  that is 4, whereas the rest of the sets from  $11 \leq l \leq 44$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in the set  $\bigcup_{u=1}^{10} R_u = \{x_1^1, x_2^1, x_3^1, x_4^1, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2\}$  and by lemma 5.1.5, taking their sum, we get

$$dim_{frac}(\mathbb{G}) = \sum_{u=1}^9 \frac{1}{4} = \frac{9}{4}.$$

(iii) When  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $n \geq 3$  and  $q_p = n + p - 1$ .

The possible local resolving neighbourhoods of  $\mathbb{G}$  are  $x_j^p x_{j+1}^p$ , and  $vx_j$ . Among these sets the sets with the minimum are  $R\{x_j^p x_{j+1}^p\}$  where  $1 \leq p \leq 2$ , that is  $|R_u| = |R\{x_j^p x_{j+1}^p\}| = n$  and  $\bigcup_{u=1}^p R_u = \{x_k^{p-1}, x_l^p | 1 \leq p \leq 2, 1 \leq k \leq p-1 \wedge 1 \leq l \leq p\}$ . Therefore, let  $\delta = |\bigcup_{u=1}^p R_u| = 2n + 1$  and  $\omega = |R_u| = n$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

. Assigning all the vertices in the set  $\bigcup_{u=1}^p R_u$  the weight  $\frac{1}{\omega}$  and by lemma 5.1.5, calculating their sum, we get

$$dim_{frac}(\mathbb{G}) = \sum_{u=1}^p \frac{1}{\omega} = \frac{2n+1}{n}.$$

**Theorem 5.4.3** Suppose  $\mathbb{G} \cong \mathbb{GWW}(m; q_1, q_2, \dots, q_m)$ , for  $m \geq 3$ ,  $n \geq 3$ ,  $1 \leq p \leq m$  and  $q_p = n + 2(p - 1)$ , then

$$dim_{frac}(\mathbb{G}) = 1.$$

**Proof:**

**Proof:**

**Case I:** For  $n \equiv 1 \pmod{2}$

(i): When  $m = 3$ ,  $q_1 = 3$ ,  $q_2 = 5$  and  $q_3 = 7$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods | Elements |
|--------------------------|----------|
|--------------------------|----------|

|                             |                                                |
|-----------------------------|------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1\}$                             |
| $R_2 = R\{x_2^1 x_3^1\}$    | $\{x_2^1, x_3^1\}$                             |
| $R_3 = R\{x_1^1 x_3^1\}$    | $\{x_1^1, x_3^1\}$                             |
| $R_4 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_5^2\}$               |
| $R_5 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2\}$               |
| $R_6 = R\{x_3^2 x_4^2\}$    | $\{x_2^2, x_3^2, x_4^2, x_5^2\}$               |
| $R_7 = R\{x_4^2 x_5^2\}$    | $\{x_1^2, x_3^2, x_4^2, x_5^2\}$               |
| $R_8 = R\{x_1^2 x_5^2\}$    | $\{x_1^2, x_2^2, x_4^2, x_5^2\}$               |
| $R_9 = R\{x_1^3 x_2^3\}$    | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_6^3, x_7^3\}$ |
| $R_{10} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_7^3\}$ |
| $R_{11} = R\{x_3^3 x_4^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\}$ |
| $R_{12} = R\{x_4^3 x_5^3\}$ | $\{x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3\}$ |
| $R_{13} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3\}$ |
| $R_{14} = R\{x_6^3 x_7^3\}$ | $\{x_1^3, x_2^3, x_4^3, x_5^3, x_6^3, x_7^3\}$ |
| $R_{15} = R\{x_1^3 x_7^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_5^3, x_6^3, x_7^3\}$ |
| $R_{16} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_3^1\}$             |
| $R_{17} = R\{vx_1^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_5^2\}$             |
| $R_{18} = R\{vx_1^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_7^3\}$             |
| $R_{19} = R\{vx_2^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_3^1\}$             |
| $R_{20} = R\{vx_2^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_5^2\}$             |
| $R_{21} = R\{vx_2^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_3^3\}$             |
| $R_{22} = R\{vx_3^1\}$      | $V(\mathbb{G}) - \{x_1^1, x_2^1\}$             |
| $R_{23} = R\{vx_3^2\}$      | $V(\mathbb{G}) - \{x_2^2, x_4^2\}$             |
| $R_{24} = R\{vx_3^3\}$      | $V(\mathbb{G}) - \{x_2^3, x_4^3\}$             |
| $R_{25} = R\{vx_4^2\}$      | $V(\mathbb{G}) - \{x_3^2, x_5^2\}$             |
| $R_{26} = R\{vx_4^3\}$      | $V(\mathbb{G}) - \{x_3^3, x_5^3\}$             |
| $R_{27} = R\{vx_5^2\}$      | $V(\mathbb{G}) - \{x_1^2, x_4^2\}$             |
| $R_{28} = R\{vx_5^3\}$      | $V(\mathbb{G}) - \{x_4^3, x_6^3\}$             |
| $R_{29} = R\{vx_6^3\}$      | $V(\mathbb{G}) - \{x_5^3, x_7^3\}$             |
| $R_{30} = R\{vx_7^3\}$      | $V(\mathbb{G}) - \{x_1^3, x_6^3\}$             |

From the above resolving neighbourhood sets, the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 3$  that is 2, whereas the rest of the sets from  $7 \leq l \leq 30$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{2}$  to the vertices in the set

$\bigcup_{u=1}^3 R_u = \bigcup_{u=1}^3 \{x_1^1, x_2^1, x_3^1\}$  and from lemma 5.1.5, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^3 \frac{1}{2} = \frac{3}{2}.$$

(ii) When  $m = 3, q_1 = 3, q_2 = 5, q_3 = 7$  and  $q_3 = 9$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods | Elements           |
|--------------------------|--------------------|
| $R_1 = R\{x_1^1 x_2^1\}$ | $\{x_1^1, x_2^1\}$ |
| $R_2 = R\{x_2^1 x_3^1\}$ | $\{x_2^1, x_3^1\}$ |
| $R_3 = R\{x_1^1 x_3^1\}$ | $\{x_1^1, x_3^1\}$ |

$$\begin{aligned}
R_4 &= R\{x_1^2 x_2^2\} & \{x_1^2, x_2^2, x_3^2, x_5^2\} \\
R_5 &= R\{x_2^2 x_3^2\} & \{x_1^2, x_2^2, x_3^2, x_4^2\} \\
R_6 &= R\{x_3^2 x_4^2\} & \{x_2^2, x_3^2, x_4^2, x_5^2\} \\
R_7 &= R\{x_4^2 x_5^2\} & \{x_1^2, x_3^2, x_4^2, x_5^2\} \\
R_8 &= R\{x_1^2 x_5^2\} & \{x_1^2, x_2^2, x_4^2, x_5^2\} \\
R_9 &= R\{x_1^3 x_2^3\} & \{x_1^3, x_2^3, x_3^3, x_4^3, x_6^3, x_7^3\} \\
R_{10} &= R\{x_2^3 x_3^3\} & \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_7^3\} \\
R_{11} &= R\{x_3^3 x_4^3\} & \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3\} \\
R_{12} &= R\{x_4^3 x_5^3\} & \{x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3\} \\
R_{13} &= R\{x_5^3 x_6^3\} & \{x_1^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3\} \\
R_{14} &= R\{x_6^3 x_7^3\} & \{x_1^3, x_2^3, x_4^3, x_5^3, x_6^3, x_7^3\} \\
R_{15} &= R\{x_1^3 x_7^3\} & \{x_1^3, x_2^3, x_3^3, x_5^3, x_6^3, x_7^3\} \\
R_{16} &= R\{x_1^4 x_2^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_7^4, x_8^4, x_9^4\} \\
R_{17} &= R\{x_2^4 x_3^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_8^4, x_9^4\} \\
R_{18} &= R\{x_3^4 x_4^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_9^4\} \\
R_{19} &= R\{x_4^4 x_5^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4\} \\
R_{20} &= R\{x_5^4 x_6^4\} & \{x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4\} \\
R_{21} &= R\{x_6^4 x_7^4\} & \{x_1^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4\} \\
R_{22} &= R\{x_7^4 x_8^4\} & \{x_1^4, x_2^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4\} \\
R_{23} &= R\{x_8^4 x_9^4\} & \{x_1^4, x_2^4, x_3^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4\} \\
R_{24} &= R\{x_1^4 x_9^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_6^4, x_7^4, x_8^4, x_9^4\} \\
R_{25} &= R\{vx_1^1\} & V(\mathbb{G}) - \{x_2^1, x_3^1\} \\
R_{26} &= R\{vx_1^2\} & V(\mathbb{G}) - \{x_2^2, x_5^2\} \\
R_{27} &= R\{vx_1^3\} & V(\mathbb{G}) - \{x_2^3, x_7^3\} \\
R_{28} &= R\{vx_1^4\} & V(\mathbb{G}) - \{x_2^4, x_9^4\} \\
R_{29} &= R\{vx_2^1\} & V(\mathbb{G}) - \{x_1^1, x_3^1\} \\
R_{30} &= R\{vx_2^2\} & V(\mathbb{G}) - \{x_1^2, x_3^2\} \\
R_{31} &= R\{vx_2^3\} & V(\mathbb{G}) - \{x_1^3, x_3^3\} \\
R_{32} &= R\{vx_2^4\} & V(\mathbb{G}) - \{x_1^4, x_3^4\} \\
R_{33} &= R\{vx_3^1\} & V(\mathbb{G}) - \{x_1^1, x_2^1\} \\
R_{34} &= R\{vx_3^2\} & V(\mathbb{G}) - \{x_2^2, x_4^2\} \\
R_{35} &= R\{vx_3^3\} & V(\mathbb{G}) - \{x_2^3, x_4^3\} \\
R_{36} &= R\{vx_3^4\} & V(\mathbb{G}) - \{x_2^4, x_4^4\} \\
R_{37} &= R\{vx_4^2\} & V(\mathbb{G}) - \{x_3^2, x_5^2\} \\
R_{38} &= R\{vx_4^3\} & V(\mathbb{G}) - \{x_3^3, x_5^3\} \\
R_{39} &= R\{vx_4^4\} & V(\mathbb{G}) - \{x_3^4, x_5^4\} \\
R_{40} &= R\{vx_5^2\} & V(\mathbb{G}) - \{x_2^2, x_4^2\} \\
R_{41} &= R\{vx_5^3\} & V(\mathbb{G}) - \{x_3^3, x_6^3\} \\
R_{42} &= R\{vx_5^4\} & V(\mathbb{G}) - \{x_4^4, x_6^4\} \\
R_{43} &= R\{vx_6^3\} & V(\mathbb{G}) - \{x_5^3, x_7^3\} \\
R_{44} &= R\{vx_6^4\} & V(\mathbb{G}) - \{x_5^4, x_7^4\} \\
R_{45} &= R\{vx_7^3\} & V(\mathbb{G}) - \{x_1^3, x_6^3\} \\
R_{46} &= R\{vx_7^4\} & V(\mathbb{G}) - \{x_6^4, x_8^4\} \\
R_{47} &= R\{vx_8^4\} & V(\mathbb{G}) - \{x_7^4, x_9^4\} \\
R_{48} &= R\{vx_9^4\} & V(\mathbb{G}) - \{x_1^4, x_8^4\}
\end{aligned}$$

As it is evident from the above resolving neighbourhood sets, that the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 3$  that is 2, whereas the rest of the sets from  $4 \leq l \leq 48$  are the sets having the higher

cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{2}$  to the vertices in the set  $\bigcup_{u=1}^3 R_u = \{x_1^1, x_2^1, x_3^1\}$  and by lemma 5.1.5, summing them all, we get

$$\dim_{frac}(\mathbb{G}) = \sum_{u=1}^3 \frac{1}{2} = \frac{3}{2}.$$

(iii) When  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $n \geq 3$ ,  $n \equiv 1 \pmod{2}$  and  $q_p = n + 2(p - 1)$ .

The local resolving neighbourhoods of  $\mathbb{G}$  are  $x_j^p x_{j+1}^p$ , and  $vx_j$ . Out of these the set with minimum cardinality is  $x_k^p x_{k+1}^p$  where  $q_p = \min\{q_p | 1 \leq p \leq m\}$ . Therefore, in this case, there will be  $q_p$  resolving neighbourhoods with minimum cardinality that are  $x_j^p x_{j+1}^p$  with  $|R\{x_j^p x_{j+1}^p\}| = n - 1$ . Therefore, let  $|q_p| = \delta$  and  $\omega = |q_p - 1|$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

. Now, after assigning the label  $\frac{1}{\omega}$  to all the vertices in the set  $\bigcup_{u=1}^{\dot{p}} R_u$  and by lemma 5.1.5, summing all the labels, we get

$$\dim_{frac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega} = \frac{n}{n-1}.$$

**Case II:** For  $n \equiv 0 \pmod{2}$

(i): When  $m = 3$ ,  $q_1 = 4$ ,  $q_2 = 6$  and  $q_3 = 8$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                                                     |
|-----------------------------|--------------------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_2 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_3 = R\{x_3^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_4 = R\{x_1^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_5 = R\{x_1^2 x_2^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_6 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_7 = R\{x_3^2 x_4^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_8 = R\{x_4^2 x_5^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_9 = R\{x_5^2 x_6^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_{10} = R\{x_1^3 x_2^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{11} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{12} = R\{x_3^3 x_4^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{13} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{14} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{15} = R\{x_6^3 x_7^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{16} = R\{x_7^3 x_8^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{17} = R\{x_1^3 x_8^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{18} = R\{x_1^3 x_8^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{19} = R\{vx_1^1\}$      | $V(\mathbb{G}) - \{x_2^1, x_4^1\}$                           |

$$\begin{array}{ll}
R_{20} = R\{vx_1^2\} & V(\mathbb{G}) - \{x_2^2, x_6^2\} \\
R_{21} = R\{vx_1^3\} & V(\mathbb{G}) - \{x_2^3, x_8^3\} \\
R_{22} = R\{vx_2^1\} & V(\mathbb{G}) - \{x_1^1, x_3^1\} \\
R_{23} = R\{vx_2^2\} & V(\mathbb{G}) - \{x_1^2, x_3^2\} \\
R_{24} = R\{vx_2^3\} & V(\mathbb{G}) - \{x_1^3, x_3^3\} \\
R_{25} = R\{vx_3^1\} & V(\mathbb{G}) - \{x_2^1, x_4^1\} \\
R_{26} = R\{vx_3^2\} & V(\mathbb{G}) - \{x_2^2, x_4^2\} \\
R_{27} = R\{vx_3^3\} & V(\mathbb{G}) - \{x_2^3, x_4^3\} \\
R_{28} = R\{vx_4^1\} & V(\mathbb{G}) - \{x_1^1, x_3^1\} \\
R_{29} = R\{vx_4^2\} & V(\mathbb{G}) - \{x_2^2, x_5^2\} \\
R_{30} = R\{vx_4^3\} & V(\mathbb{G}) - \{x_3^3, x_5^3\} \\
R_{31} = R\{vx_5^2\} & V(\mathbb{G}) - \{x_4^2, x_6^2\} \\
R_{32} = R\{vx_5^3\} & V(\mathbb{G}) - \{x_4^3, x_6^3\} \\
R_{33} = R\{vx_6^2\} & V(\mathbb{G}) - \{x_1^2, x_5^2\} \\
R_{34} = R\{vx_6^3\} & V(\mathbb{G}) - \{x_3^3, x_7^3\} \\
R_{35} = R\{vx_7^3\} & V(\mathbb{G}) - \{x_6^3, x_8^3\} \\
R_{36} = R\{vx_8^3\} & V(\mathbb{G}) - \{x_1^3, x_7^3\}
\end{array}$$

From the above resolving neighbourhood sets, the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 4$  that is 4, whereas the rest of the sets from  $5 \leq l \leq 36$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{4}$  to the vertices in the set

$$\bigcup_{u=1}^4 R_u = \bigcup_{u=1}^4 \{x_1^1, x_2^1, x_3^1, x_4^1\} \text{ and from lemma 5.1.5, summing them all, we get}$$

$$dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^4 \frac{1}{4} = \frac{4}{4} = 1.$$

(ii) When  $m = 4, q_1 = 4, q_2 = 6, q_3 = 8$  and  $q_4 = 10$ .

The required local resolving neighbourhoods are given by:

| Resolving Neighbourhoods    | Elements                                                     |
|-----------------------------|--------------------------------------------------------------|
| $R_1 = R\{x_1^1 x_2^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_2 = R\{x_2^1 x_3^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_3 = R\{x_3^1 x_4^1\}$    | $\{x_1^1, x_2^1, x_3^1, x_4^1\}$                             |
| $R_4 = R\{x_1^2 x_2^2\}$    | $\{x_1^1, x_2^2, x_3^1, x_4^1\}$                             |
| $R_5 = R\{x_1^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_6 = R\{x_2^2 x_3^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_7 = R\{x_3^2 x_4^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_8 = R\{x_4^2 x_5^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_9 = R\{x_5^2 x_6^2\}$    | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_{10} = R\{x_1^3 x_2^3\}$ | $\{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2\}$               |
| $R_{11} = R\{x_1^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{12} = R\{x_2^3 x_3^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{13} = R\{x_3^3 x_4^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{14} = R\{x_4^3 x_5^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{15} = R\{x_5^3 x_6^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |
| $R_{16} = R\{x_6^3 x_7^3\}$ | $\{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\}$ |

$$\begin{aligned}
R_{17} &= R\{x_7^3 x_8^3\} & \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\} \\
R_{18} &= R\{x_1^3 x_8^3\} & \{x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, x_6^3, x_7^3, x_8^3\} \\
R_{19} &= R\{x_1^4 x_2^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{20} &= R\{x_2^4 x_3^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{21} &= R\{x_3^4 x_4^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{22} &= R\{x_4^4 x_5^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{23} &= R\{x_5^4 x_6^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{24} &= R\{x_6^4 x_7^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{25} &= R\{x_7^4 x_8^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{26} &= R\{x_8^4 x_9^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{27} &= R\{x_9^4 x_{10}^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{28} &= R\{x_1^4 x_{10}^4\} & \{x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4, x_8^4, x_9^4, x_{10}^4\} \\
R_{29} &= R\{vx_1^1\} & V(\mathbb{G}) - \{x_2^1, x_4^1\} \\
R_{30} &= R\{vx_1^2\} & V(\mathbb{G}) - \{x_2^2, x_6^2\} \\
R_{31} &= R\{vx_1^3\} & V(\mathbb{G}) - \{x_2^3, x_8^3\} \\
R_{32} &= R\{vx_1^4\} & V(\mathbb{G}) - \{x_2^4, x_{10}^4\} \\
R_{33} &= R\{vx_2^1\} & V(\mathbb{G}) - \{x_1^1, x_3^1\} \\
R_{34} &= R\{vx_2^2\} & V(\mathbb{G}) - \{x_1^2, x_3^2\} \\
R_{35} &= R\{vx_2^3\} & V(\mathbb{G}) - \{x_1^3, x_3^3\} \\
R_{36} &= R\{vx_2^4\} & V(\mathbb{G}) - \{x_1^4, x_3^4\} \\
R_{37} &= R\{vx_3^1\} & V(\mathbb{G}) - \{x_2^1, x_4^1\} \\
R_{38} &= R\{vx_3^2\} & V(\mathbb{G}) - \{x_2^2, x_4^2\} \\
R_{39} &= R\{vx_3^3\} & V(\mathbb{G}) - \{x_2^3, x_4^3\} \\
R_{40} &= R\{vx_3^4\} & V(\mathbb{G}) - \{x_2^4, x_4^4\} \\
R_{41} &= R\{vx_4^1\} & V(\mathbb{G}) - \{x_1^1, x_3^1\} \\
R_{42} &= R\{vx_4^2\} & V(\mathbb{G}) - \{x_2^2, x_5^2\} \\
R_{43} &= R\{vx_4^3\} & V(\mathbb{G}) - \{x_3^3, x_5^3\} \\
R_{44} &= R\{vx_4^4\} & V(\mathbb{G}) - \{x_3^4, x_5^4\} \\
R_{45} &= R\{vx_5^2\} & V(\mathbb{G}) - \{x_2^2, x_6^2\} \\
R_{46} &= R\{vx_5^3\} & V(\mathbb{G}) - \{x_3^3, x_6^3\} \\
R_{47} &= R\{vx_5^4\} & V(\mathbb{G}) - \{x_4^4, x_6^4\} \\
R_{48} &= R\{vx_6^2\} & V(\mathbb{G}) - \{x_1^2, x_5^2\} \\
R_{49} &= R\{vx_6^3\} & V(\mathbb{G}) - \{x_5^3, x_7^3\} \\
R_{50} &= R\{vx_6^4\} & V(\mathbb{G}) - \{x_5^4, x_7^4\} \\
R_{51} &= R\{vx_7^3\} & V(\mathbb{G}) - \{x_6^3, x_8^3\} \\
R_{52} &= R\{vx_7^4\} & V(\mathbb{G}) - \{x_6^4, x_8^4\} \\
R_{53} &= R\{vx_8^3\} & V(\mathbb{G}) - \{x_1^3, x_7^3\} \\
R_{54} &= R\{vx_8^4\} & V(\mathbb{G}) - \{x_7^4, x_9^4\} \\
R_{55} &= R\{vx_9^4\} & V(\mathbb{G}) - \{x_8^4, x_{10}^4\} \\
R_{56} &= R\{vx_{10}^4\} & V(\mathbb{G}) - \{x_1^4, x_9^4\}
\end{aligned}$$

As it is evident from the above resolving neighbourhood sets, that the sets with the minimum cardinality are  $R_l$  for  $1 \leq l \leq 4$  that is 4, whereas the rest of the sets from  $5 \leq l \leq 56$  are the sets having the higher cardinalities. Therefore, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that assigning  $\frac{1}{6}$  to the vertices in

the set  $\bigcup_{u=1}^4 R_u = \bigcup_{u=1}^4 \{x_1^1, x_2^1, x_3^1, x_4^1\}$  and by lemma 5.1.5, summing them all, we get

$$\dim_{lfrac}(\mathbb{G}) = \sum_{u=1}^4 \frac{1}{4} = \frac{4}{4} = 1.$$

(iii) When  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $n \geq 3$ ,  $n \equiv 1 \pmod{2}$  and  $q_p = n + 2(p - 1)$ .

The local resolving neighbourhoods of  $\mathbb{G}$  are  $x_j^p x_{j+1}^p$ , and  $vx_j$ . Among them the set with minimum cardinality is  $x_k^p x_{k+1}^p$  where  $q_p = \min\{q_p | 1 \leq p \leq m\}$ . Therefore, in this case, there will be  $q_p$  resolving neighbourhoods with minimum cardinality that are  $x_j^p x_{j+1}^p$  with  $|R\{x_j^p x_{j+1}^p\}| = n$ . Therefore, let  $|\bigcup_{u=1}^p R_u| = n = \delta$  and  $\omega = |R_u| = n$ . Now, we define a mapping  $\psi : V(\mathbb{G}) \rightarrow [0, 1]$ , such that

$$\phi(v) = \begin{cases} \frac{1}{\omega} & \text{for } v \in \bigcup_{u=1}^m R_u \\ 0 & \text{for } v \in V(\mathbb{G}) - \bigcup_{u=1}^m R_u \end{cases}$$

. Now, after assigning the label  $\frac{1}{\omega}$  to all the vertices in the set  $\bigcup_{u=1}^p R_u$  and by lemma 5.1.5, summing all the labels, we get

$$\dim_{lfac}(\mathbb{G}) = \sum_{u=1}^{\delta} \frac{1}{\omega} = \frac{n}{n} = 1.$$

**Theorem 5.4.4** Let  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $q_1 \geq q_2 \geq \dots \geq q_p$  and  $\mathbb{G} \cong \text{GDW}(m; q_1, q_2, \dots, q_p)$ . Then  $\dim_{lfac}(\mathbb{G}) \in [1, m]$ .

**Proof:**

**Case I:** (i) When  $q_1 = q_2 = \dots = q_p$  and  $q_p \equiv 0 \pmod{2}$ .

This case will be trivial as the graph so formed will be a bipartite graph. Thus

$$\dim_{lfac}(\mathbb{G}) = 1.$$

(ii) When  $q_1 \geq q_2 \geq \dots \geq q_p$ .

As we have proved in theorem 5.2.2 and 5.2.3 that when all  $q_1, q_2, \dots, q_p$  are greater or equal to each other, thus for this case the minimum resolving sets from the cycle having the minimum length among all the cycles will be considered. Also, from Lemma 5.1.3, we have found that  $\dim_{lfac}$  will be equal to the summation of all the labels assigned to all the vertices of the minimum resolving sets. Hence,

$$\dim_{lfac}(\mathbb{G}) = 1.$$

**Case II:** When  $q_1 = q_2 = \dots = q_p$ . As we have proved in theorem 5.2.1(a) that when all  $q_1, q_2, \dots, q_p$  are equal, the minimum resolving sets in this case will be equal to the one less than the length of each cycle. Moreover, we have proved in Lemma 5.1.3, that it  $\dim_{lfac}$  will be equal to the summation of all the labels assigned to all the vertices of  $\mathbb{G}$  excluding  $v$ . Therefore,

$$\dim_{lfac}(\mathbb{G}) = m.$$

From the cases mentioned above, we have found that  $\dim_{lfac}(\mathbb{G}) \in [1, m]$ .

**Theorem 5.4.5** Let  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $q_1 \geq q_2 \geq \dots \geq q_p$ ,  $q_p \geq 6$  and  $\mathbb{G} \cong \text{GPW}(m; q_1, q_2, \dots, q_p)$ . Then  $\dim_{lfac}(\mathbb{G}) = 2m$ .

**Proof:** as we can see from Theorem 5.3.1 and 5.3.2 that all the minimum resolving sets in each case are equal to  $R\{x_1^p x_2^p\} = \{x_1^p, x_2^p, x_3^p\}$  and  $R\{x_{n-1}^p x_n^p\} = \{x_{n-2}^p, x_{n-1}^p, x_n^p\}$ . Also  $\bigcup_{u=1}^p R_u = \bigcup_{u=1}^p \{x_1^p, x_2^p, x_3^p, x_{n-2}^p, x_{n-1}^p, x_n^p\}$ . Thus assigning the labels and summing them all, we get

$$\dim_{\text{lfrc}} \mathbb{G} = 2m.$$

**Theorem 5.4.6** Let  $m \geq 3$ ,  $1 \leq p \leq m$ ,  $q_1 \geq q_2 \geq \dots \geq q_p$  and  $\mathbb{G} \cong \mathbb{G}^{\text{WW}}(m; q_1, q_2, \dots, q_p)$ . Then  $\dim_{\text{lfrc}}(\mathbb{G}) \in [1, m]$ .

**Proof:**

**Case I:** (i) When  $q_1 \geq q_2 \geq \dots \geq q_p$ .

As we have proved in theorem 5.4.3 that when all  $q_1, q_2, \dots, q_p$  are greater or equal to each other, thus for this case the minimum resolving sets from the wheel having the minimum spokes regardless of even or odd length. Thus, we will consider the minimum resolving set. Also, from Lemma 5.1.5, we have found that  $\dim_{\text{lfrc}}$  will be equal to the summation of all the labels assigned to all the vertices of the minimum resolving sets. Hence,

$$\dim_{\text{lfrc}}(\mathbb{G}) = 1.$$

**Case II:** When  $q_1 = q_2 = \dots = q_p$ . As we have proved in theorem 5.2.1(a) that when all  $q_1, q_2, \dots, q_p$  are equal, the minimum resolving sets in this case will be equal to resolving sets of cycles in each  $q_p$ . Earlier, we have proved in Lemma 5.1.4, that it  $\dim_{\text{lfrc}}$  will be equal to the summation of all the labels assigned to all the vertices of  $\mathbb{G}$  excluding  $v$ . Therefore,

$$\dim_{\text{lfrc}}(\mathbb{G}) = m.$$

As we can see from the above discussion, that is

$$\dim_{\text{lfrc}}(\mathbb{G}) \in [1, m].$$

## 6. Conclusion

We close our discussion by the following remarks:

- The LFMD of  $C_m$  for  $m \equiv 1 \pmod{2}$  is  $\frac{m}{m-1}$  and 1 when  $m \equiv 0 \pmod{2}$ .
- The LFMD of  $\mathbb{G}_I$  is  $\frac{s^3 - 5s^2 + 9s - 6}{4s - 7}$ .
- The LFMD of  $\mathbb{G}_{II}$  is  $\frac{s^3 - 6s^2 + 10s - 3}{4s + 7}$ .
- The LFMD of  $\mathbb{G}^{\text{PW}}(m; q_1, q_2, \dots, q_p)$  is  $2m$ .
- The LFMD of  $\mathbb{G}^{\text{DW}}(m; q_1, q_2, \dots, q_p)$  and  $\mathbb{G}^{\text{WW}}(m; q_1, q_2, \dots, q_p)$  lies in the interval  $[1, m]$ .
- As shown in Figure 6.1, the LFMDs of the graphs under consideration are shown. As its evident from the 3D plot illustration that the dominant LFMD among all the graphs is of  $C_m$ .

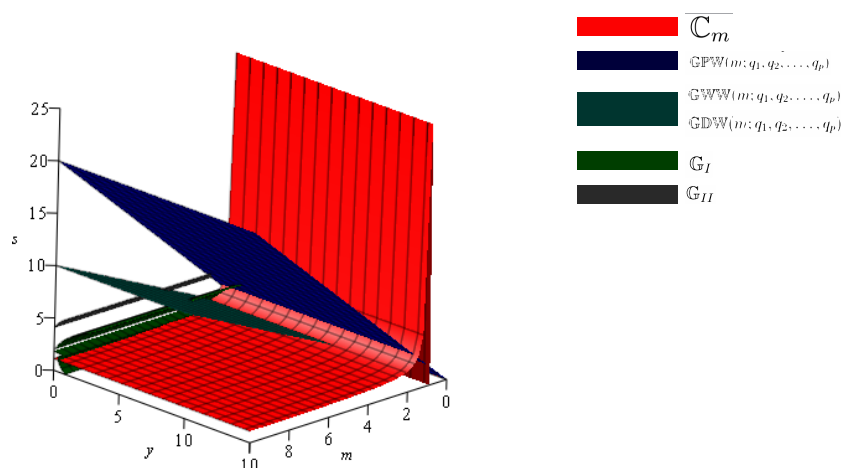


**Figure 6.1:** Behaviour of the obtained results as  $n \rightarrow \infty$

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