



SEM-deficiency of Union of Subdivided Stars

Abdul Raheem¹, Roslan Hasni² and Sadia Riaz^{3,*}

¹Higher Education Department, Govt. Graduate College Asghar Mall Rawalpindi, Pakistan;,
drabdurraheem@gpgcam.edu.pk

²Faculty of Ocean Engineering technology and Informatics, University Malaysia Terengganu, Malaysia;,
hroslan@umt.edu.my

³Department of Mathematics,, National University of Modern Languages, Rawalpindi, Pakistan;,
sadia.riaz@numl.edu.pk

* Corresponding Author: [e-mail: sadia.riaz@numl.edu.pk]

Abstract

An edge-magic total labeling of a connected graph $\Gamma = (V_\Gamma, E_\Gamma)$ is a bijective map $\pi : V_\Gamma \cup E_\Gamma \rightarrow \{1, 2, \dots, |V_\Gamma| + |E_\Gamma|\}$ such that there exists a fixed number (say d) under the condition of $\pi(s) + \pi(st) + \pi(t) = d$ for all $st \in E_\Gamma$. In addition, if we label the vertices from $\{1, 2, \dots, |V_\Gamma|\}$, then this labeling is called super edge-magic total labeling and the graph Γ admitting this labeling is super edge-magic total graph. The super edge-magic deficiency (SEM-deficiency) $v_s(\Gamma)$ of a graph Γ is a minimum non-negative number m if $\Gamma \cup mK_1$ is a super edge-magic total graph or ∞ if there does not exist m . In the current paper, we study the super edge-magic deficiency of forest of two non-isomorphic copies of the subdivided stars.

Keywords: Magic deficiency; Trees; Operation of subdivision

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1. Introduction and Preliminaries

A connected (p, q) -graph $\Gamma = (V_\Gamma, E_\Gamma)$ of order p and size q is said to be an edge-magic total if it admits a bijective map $\pi : V_\Gamma \cup E_\Gamma \rightarrow \{1, 2, \dots, |V_\Gamma| + |E_\Gamma|\}$ such that we get the edge-weights $\pi(s) + \pi(st) + \pi(t) = d$, where $st \in E_\Gamma$ for all edges and d is constant integer. If we assign the smallest labels to the vertices then an edge-magic total labeling is called a super edge magic total labeling. Moreover, a graph is called a super edge magic total graph if it admits super edge magic total labeling. The super edge-magic deficiency of a graph Γ is a non-negative integer m if the graph $\Gamma \cup mK_1$ admits super edge-magic total labeling. For more detail about basic notations and terminologies that are used in the rest of the paper, we refer to [16].

Firstly, the concept of edge-magic total was appeared in 1970 by the work of Kotzig et al.[12]. They also proved that for each graph Γ , there must exist edge-magic total graph $H = \Gamma \cup mK_1$, where m is non-negative integer. By using this fact, we get the concept of edge-magic deficiency of Γ that is $\Gamma \cup mK_1$ is called edge-magic and denoted by $v(\Gamma)$. Particularly, we can write

$$v(\Gamma) = \min\{m \geq 0 : \Gamma \cup mK_1 \text{ is an edge-magic total graph}\}.$$

In the same article, Kotzig et al. gave the following upper bound of the edge-magic deficiency of Γ having m vertices:

$$v(\Gamma) \leq F_{m+2} - 2 - m - \frac{m(m-1)}{2},$$

where F_m is m th Fibonacci number. Later on, Figueroa-Centeno et al. [6] defined the super edge-magic deficiency $v_s(\Gamma)$, for a graph Γ as the minimum non-negative integer m if $\Gamma \cup mK_1$ admits super edge-magic total labeling, or ∞ if no such integer m exist. we can write precisely that, if

$$M(\Gamma) = \{m \geq 0 : \Gamma \cup mK_1 \text{ is super edge-magic total graph}\},$$

then

$$v_s(\Gamma) = \begin{cases} \min(M(\Gamma)), & \text{if } M(\Gamma) \neq \emptyset, \\ \infty, & \text{if } M(\Gamma) = \emptyset, \end{cases}$$

For every graph, we can easily check that for graph Γ , $v(\Gamma) \leq v_s(\Gamma)$. In [6, 8], Figueroa-Centeno et al. computed the exact values of several graphs for super-edge magic deficiencies including cycles, complete, 2-regular and complete bipartite graphs. They proved that deficiency is finite for all forest. Particularly, they found that

$$v_s(mK_2) = \begin{cases} 1, & \text{if } n \equiv 0(\text{mod}2), \\ 0, & \text{if } n \equiv 1(\text{mod}2), \end{cases}$$

Ngurah et al.[14] found some upper bound of fans, double fans and wheels for the super edge-magic deficiency. In [7] Figueroa-Centeno et al. presented

$$v_s(P_{n_1} \cup K_{1,n_2}) = \begin{cases} 1, & \text{if } n_1 = 2 \text{ and } n \equiv 1(\text{mod}2) \text{ or} \\ & \text{if } n_1 = 3 \text{ and } n \equiv 0(\text{mod}3) \\ 0, & \text{otherwise,} \end{cases}$$

They also proved in same paper that

$$v_s(K_{1,n_1} \cup K_{1,n_2}) = \begin{cases} 0, & \text{if } n_1 \text{ multiple of } n_1 + 1 \text{ or,} \\ & \text{if } n_2 \text{ multiple of } n_2 + 1, \\ 1, & \text{otherwise,} \end{cases}$$

Figueroa-centeno et al. was given conjecture that all forest having two components admit super edge-magic deficiency less than or equal to 1. Baig et al. [3, 4] determined the edge-magic deficiency of star forest and also for find deficiencies for web $Wb_{n,n}$, Jahangir $J_{2,n}$ graphs, crown products $L_n \odot K_1$, $K_4 \odot nK_1$. Santhosh et al. [15] showed that $v(K_2 \odot C_n) \leq \frac{n-3}{2}$, for n odd at least 3. For more details about the latest results on labeling of graphs, see [1, 9, 17, 18]. Moreover, for further study, reader may consult [2, 10, 11]. In the present paper, we compute the super edge-magic deficiency of union of non-isomorphic subdivision of star $\mu_s(\underbrace{Sb(l, l, \dots, l)}_{(n-1)\text{-times}} \cup \underbrace{Sb(l, l, \dots, l)}_{n\text{-times}}) \leq 2n - 1$.

In order to compute the main results which presented in this paper, we shall use frequently following Lemma.

Lemma 1.1.[5] The necessary and sufficient condition for (p, q) -graph Γ admits super edge-magic total, such that a bijective map $\pi : V_\Gamma \rightarrow \{1, 2, \dots, |V_\Gamma|\}$ then $\{\pi(u) + \pi(v) | uv \in E_\Gamma \text{ consisting of } q \text{ consecutive integers. In such a case, } \pi \text{ expanded to a super edge-magic total.}$

Now, we define the subdivision of a star as follows;

Definition 1.2. A subdivision of star $\Gamma \cong Sb(l_1, l_2, \dots, l_n)$ is a tree obtained by putting $l_j - 1$ vertices to all j th edge of the star $K_{1,n}$ for $l_i \geq 1$, $1 \leq j \leq n$, and $n \geq 3$. Let us suppose vertex-set and edge-set as;
 $V(\Gamma) = \{l_0\} \cup \{l_{ij} : 1 \leq i \leq n ; 1 \leq j \leq l_i\}$ and $E(\Gamma) = \{l_0 l_{i1} : 1 \leq i \leq n\} \cup \{l_{ij} l_{i(j+1)} : 1 \leq i \leq n ; 1 \leq j \leq l_i - 1\}$.

Definition 1.3. The union of any graphs $\Gamma^1 = (V_\Gamma^1, E_\Gamma^1)$ and $\Gamma^2 = (V_\Gamma^2, E_\Gamma^2)$ is a graph $\Gamma = \Gamma^1 \cup \Gamma^2$ with vertex-set $V_\Gamma = V_\Gamma^1 \cup V_\Gamma^2$ and the edge-set $E_\Gamma^1 = E_\Gamma^1 \cup E_\Gamma^2$.

2. Main Results

This section deals the main theorems, in which compute the upper bound for the super edge-magic deficiency of $Sb(\underbrace{l, l, \dots, l}_{(n-1)\text{-times}}) \cup Sb(\underbrace{l, l, \dots, l}_{n\text{-times}})$, Where $n \geq 3$ and $l \geq 2$.

Theorem 2.1. Prove that graph $\Gamma \cong Sb(l, l) \cup Sb(l, l, l)$ admits super edge-magic deficiency at most 4, where $l \geq 1$.

Proof. Let us suppose graph $H \cong Sb(l, l) \cup Sb(l, l, l) \cup K_1$. Let the sets of vertices and edges are presented as

$$V(H) = \{l_t : 1 \leq t \leq 2\} \cup \{l_{ij}^t : 1 \leq t \leq 2, 1 \leq i \leq 1+t, 1 \leq j \leq l\} \cup \{z_k : 1 \leq k \leq 4\}$$

and

$$E(H) = \{l_t l_{i1}^t : 1 \leq t \leq 2, 1 \leq i \leq t+1\} \cup \{l_{ij}^t l_{i(j+1)}^t : 1 \leq t \leq 2, 1 \leq i \leq t+1, 1 \leq j \leq l-1\}.$$

The cardinality of the vertices and edges are $p = 5l + 6$, and $q = 5l$.

In order to prove that H is super edge-magic total, there arises two cases:

Case-1: When $l \equiv 1 \pmod{2}$

The vertex labeling presented $\pi : V(H) \rightarrow \{1, 2, \dots, 5l + 6\}$ as given below:

$$\pi(l_t) = 5(l-t) + 11, \text{ and } \pi(z_k) = 5l + 1 + k; 1 \leq k \leq 4$$

When $1 \leq j \leq l$:

For $j \equiv 1 \pmod{2}$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{5}{2}(l-j) + 1, & \text{when } i = 1, \\ \frac{5}{2}(l-j) + 2, & \text{when } i = 2, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{5}{2}(l-j) + 4, & \text{when } i = 1, \\ \frac{5}{2}(l-j) + 5, & \text{when } i = 2, \\ \frac{3}{2}(l-j) + 3, & \text{when } i = 3, \end{cases}$$

When $j \equiv 0 \pmod{2}$:

For $1 \leq j < l-1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{5}{2}(2l-j) + 5, & \text{when } i = 1, \\ \frac{5}{2}(2l-j) + 3, & \text{when } i = 2, \end{cases}$$

$$\pi(l_{ij}^2) = \begin{cases} \frac{5}{2}(2l-j)+4, & \text{when } i=1, \\ \frac{5}{2}(2l-j)+2, & \text{when } i=2, \\ \frac{5}{2}(2l-j)+1, & \text{when } i=3, \end{cases}$$

For $j = l-1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{5l+11}{2}, & \text{when } i=1, \\ \frac{5l+13}{2}, & \text{when } i=2, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{5l+7}{2}, & \text{when } i=1, \\ \frac{5l+9}{2}, & \text{when } i=2, \\ \frac{3l+15}{2}, & \text{when } i=3, \end{cases}$$

Case-2: When $l \equiv 0(\text{mod } 2)$

$$\pi(l_t) = \frac{5(l-2t)}{2} + 11; 1 \leq t \leq 2, \text{ and } \pi(z_k) = \frac{5l}{2} + k; 1 \leq k \leq 4$$

When $1 \leq j \leq l$;

For $j \equiv 1(\text{mod } 2)$

$$\pi(l_{ij}^1) = \begin{cases} \frac{5(2l-j)+9}{2}, & \text{when } i=1, \\ \frac{5(2l-j)+11}{2}, & \text{when } i=2, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{5(2l-j)+13}{2}, & \text{when } i=1, \\ \frac{5(2l-j)+15}{2}, & \text{when } i=2, \\ \frac{5(2j-j)+17}{2}, & \text{when } i=3, \end{cases}$$

When $j \equiv 0(\text{mod } 2)$;

For $1 \leq j \leq 1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{5}{2}(l-j)+4, & \text{when } i=1, \\ \frac{5}{2}(l-j)+5, & \text{when } i=2, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{5}{2}(l-j) + 1, & \text{when } i = 1, \\ \frac{5}{2}(l-j) + 2, & \text{when } i = 2, \\ \frac{3}{2}(l-j) + 3, & \text{when } i = 3, \end{cases}$$

Thus the labeling π constitute the following set of edge-sum as given below;

$$\text{edge-sum} = \begin{cases} \{\frac{5l+13}{2}, \frac{5l+15}{2}, \dots, \frac{15l+11}{2}\}, & \text{when } l \equiv 1(\text{mod } 2), \\ \{\frac{5l+20}{2}, \frac{5l+22}{2}, \dots, \frac{15l+8}{2}\}, & \text{when } l \equiv 0(\text{mod } 2), \end{cases}$$

Thus, by Lemma 1.2, we expand π to a super edge-magic total labeling of H □

Theorem 2.2. Prove that graph $\Gamma \cong Sb(l, l, l) \cup Sb(l, l, l, l)$ admits super edge-magic deficiency at most 6, where $l \geq 1$.

Proof. Consider the graph $H \cong Sb(l, l, l) \cup Sb(l, l, l, l) \cup 6K_1$. The vertex-set and edge-set are defined as $V(H) = \{l_t : 1 \leq t \leq 2\} \cup \{l_{ij}^t : 1 \leq t \leq 2, 1 \leq i \leq 2+t, 1 \leq j \leq l\} \cup \{z_k : 1 \leq k \leq 6\}$

and

$$E(H) = \{l_t l_{i1}^t : 1 \leq t \leq 2, 1 \leq i \leq t+2\} \cup \{l_{ij}^t l_{i(j+1)}^t : 1 \leq t \leq 2, 1 \leq i \leq t+2, 1 \leq j \leq l-1\}.$$

Let us consider the cardinality of the sets of vertices and edges are $p = 7l + 8$, and $q = 7l$. We want to prove that H is super edge-magic total. For this we have two cases:

Case-1: When $l \equiv 1(\text{mod } 2)$

The scheme of vertex labeling is defined $\pi : V(H) \rightarrow \{1, 2, \dots, 7l + 8\}$ as below:

$$\pi(l_t) = 7(l-t) + 15; 1 \leq t \leq 2, \text{ and } \pi(z_k) = 7l + 1 + k; 1 \leq k \leq 6$$

When $1 \leq j \leq l$:

For $j \equiv 1(\text{mod } 2)$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{7}{2}(l-j) + 1, & \text{when } i = 1, \\ \frac{7}{2}(l-j) + 2, & \text{when } i = 2, \\ \frac{7}{2}(l-j) + 3, & \text{when } i = 3, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{7}{2}(l-j) + 5, & \text{when } i = 1, \\ \frac{7}{2}(l-j) + 6, & \text{when } i = 2, \\ \frac{7}{2}(l-j) + 7, & \text{when } i = 3, \\ \frac{7}{2}(l-j) + 4, & \text{when } i = 4, \end{cases}$$

When $j \equiv 0(\text{mod } 2)$:

For $1 \leq j < l-1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{7}{2}(2l-j)+7, & \text{when } i=1, \\ \frac{7}{2}(2l-j)+5, & \text{when } i=2, \\ \frac{7}{2}(2l-j)+3, & \text{when } i=3, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{7}{2}(2l-j)+6, & \text{when } i=1, \\ \frac{7}{2}(2l-j)+4, & \text{when } i=2, \\ \frac{7}{2}(2l-j)+2, & \text{when } i=3, \\ \frac{7}{2}(2l-j)+1, & \text{when } i=4, \end{cases}$$

For $j = l-1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{7l+15}{2}, & \text{when } i=1, \\ \frac{7l+17}{2}, & \text{when } i=2, \\ \frac{7l+19}{2}, & \text{when } i=3, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{7l+9}{2}, & \text{when } i=1, \\ \frac{7l+11}{2}, & \text{when } i=2, \\ \frac{7l+13}{2}, & \text{when } i=3, \\ \frac{7l+21}{2}, & \text{when } i=4, \end{cases}$$

Case-2: When $l \equiv 0 \pmod{2}$

$$\pi(l_t) = \frac{7(l-2t)}{2} + 15; 1 \leq t \leq 2, \text{ and } \pi(z_k) = \frac{7l}{2} + k; 1 \leq k \leq 6$$

When $1 \leq j \leq l$;

For $j \equiv 1 \pmod{2}$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{7(2l-j)+11}{2}, & \text{when } i=1, \\ \frac{7(2l-j)+13}{2}, & \text{when } i=2, \\ \frac{7(2l-j)+15}{2}, & \text{when } i=3, \end{cases}$$

$$\pi(l_{ij}^2) = \begin{cases} \frac{7(2l-j)+17}{2}, & \text{when } i = 1, \\ \frac{7(2l-j)+19}{2}, & \text{when } i = 2, \\ \frac{7(2j-j)+21}{2}, & \text{when } i = 3, \\ \frac{7(2l-j)+23}{2}, & \text{when } i = 4, \end{cases}$$

When $j \equiv 0(\text{mod } 2)$;

For $1 \leq j \leq 1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{7}{2}(l-j) + 5, & \text{when } i = 1, \\ \frac{7}{2}(l-j) + 6, & \text{when } i = 2, \\ \frac{7}{2}(l-j) + 7, & \text{when } i = 3, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{7}{2}(l-j) + 1, & \text{when } i = 1, \\ \frac{7}{2}(l-j) + 2, & \text{when } i = 2, \\ \frac{7}{2}(l-j) + 3, & \text{when } i = 3, \\ \frac{7}{2}(l-j) + 4, & \text{when } i = 4 \end{cases}$$

Thus, the labeling π constitute the following set of edge-sum as given below;

$$\text{edge-sum} = \begin{cases} \left\{ \frac{7l+17}{2}, \frac{7l+19}{2}, \dots, \frac{21l+15}{2} \right\}, & \text{when } l \equiv 1(\text{mod } 2), \\ \left\{ \frac{6l+26}{2}, \frac{7l+28}{2}, \dots, \frac{21l+24}{2} \right\}, & \text{when } l \equiv 0(\text{mod } 2), \end{cases}$$

Therefore, by Lemma 1.2, we expand π to a super edge-magic total labeling of H □

Theorem 2.3. Prove that graph $\Gamma \cong Sb(l, l, l, l) \cup Sb(l, l, l, l)$ admits super edge-magic deficiency at most 8, where $l \geq 1$.

Proof. Consider the graph $H \cong Sb(l, l, l, l) \cup Sb(l, l, l, l) \cup 8K_1$. The vertex-set and edge-set are defined as

$$V(H) = \{l_t : 1 \leq t \leq 2\} \cup \{l_{ij}^t : 1 \leq t \leq 2, 1 \leq i \leq t+3, 1 \leq j \leq l\} \cup \{z_k : 1 \leq k \leq 8\}$$

and

$$E(H) = \{l_t l_{i1}^t : 1 \leq t \leq 2, 1 \leq i \leq t+3\} \cup \{l_{ij}^t l_{i(j+1)}^t : 1 \leq t \leq 2, 1 \leq i \leq t+3, 1 \leq j \leq l-1\}.$$

The cardinality of the vertex-set and edge-set are $p = 9l + 8$, and $q = 9l$. In order to compute that H is a super edge-magic total,

Case-1: When $l \equiv 1(\text{mod } 2)$

Now, we define the labeling $\pi : V(H) \rightarrow \{1, 2, \dots, 9l + 10\}$ as given below:

$$\pi(l_t) = 9(l-t) + 19; 1 \leq t \leq 2, \text{ and } \pi(z_k) = 9l + 1 + k; 1 \leq k \leq 8$$

When $1 \leq j \leq l$:

For $j \equiv 1(\text{mod } 2)$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{9}{2}(l-j) + 1, & \text{when } i = 1, \\ \frac{9}{2}(l-j) + 2, & \text{when } i = 2, \\ \frac{9}{2}(l-j) + 3, & \text{when } i = 3, \\ \frac{9}{2}(l-j) + 4, & \text{when } i = 4, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{7}{2}(l-j) + 6, & \text{when } i = 1, \\ \frac{7}{2}(l-j) + 7, & \text{when } i = 2, \\ \frac{7}{2}(l-j) + 8, & \text{when } i = 3, \\ \frac{7}{2}(l-j) + 9, & \text{when } i = 4, \\ \frac{9}{2}(l-j) + 5, & \text{when } i = 5, \end{cases}$$

When $j \equiv 0(\text{mod } 2)$: For, $1 \leq j < l-1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{9}{2}(2l-j) + 9, & \text{when } i = 1, \\ \frac{9}{2}(2l-j) + 7, & \text{when } i = 2, \\ \frac{9}{2}(2l-j) + 5, & \text{when } i = 3, \\ \frac{9}{2}(2l-j) + 3, & \text{when } i = 4, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{9}{2}(2l-j) + 8, & \text{when } i = 1, \\ \frac{9}{2}(2l-j) + 6, & \text{when } i = 2, \\ \frac{9}{2}(2l-j) + 4, & \text{when } i = 3, \\ \frac{9}{2}(2l-j) + 2, & \text{when } i = 4, \\ \frac{9}{2}(2l-j) + 1, & \text{when } i = 5, \end{cases}$$

For $j = l-1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{9l+19}{2}, & \text{when } i = 1, \\ \frac{9l+21}{2}, & \text{when } i = 2, \\ \frac{9l+23}{2}, & \text{when } i = 3, \\ \frac{9l+25}{2}, & \text{when } i = 4, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{9l+11}{2}, & \text{when } i = 1, \\ \frac{9l+13}{2}, & \text{when } i = 2, \\ \frac{9l+15}{2}, & \text{when } i = 3, \\ \frac{9l+17}{2}, & \text{when } i = 4, \\ \frac{9l+27}{2}, & \text{when } i = 5, \end{cases}$$

Case-2: When $l \equiv 0(\text{mod } 2)$

$$\pi(l_t) = \frac{9(l-2t)}{2} + 19; 1 \leq t \leq 2, \text{ and } \pi(z_k) = \frac{9l}{2} + k; 1 \leq k \leq 8.$$

When $1 \leq j \leq l$;

For $j \equiv 1(\text{mod } 2)$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{9(2l-j)+13}{2}, & \text{when } i = 1, \\ \frac{9(2l-j)+15}{2}, & \text{when } i = 2, \\ \frac{9(2l-j)+17}{2}, & \text{when } i = 3, \\ \frac{9(2l-j)+19}{2}, & \text{when } i = 4, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{9(2l-j)+21}{2}, & \text{when } i = 1, \\ \frac{9(2l-j)+23}{2}, & \text{when } i = 2, \\ \frac{9(2j-j)+25}{2}, & \text{when } i = 3, \\ \frac{9(2l-j)+27}{2}, & \text{when } i = 4, \\ \frac{9(2l-j)+29}{2}, & \text{when } i = 5, \end{cases}$$

When $j \equiv 0(\text{mod } 2)$:

For $1 \leq j \leq 1$;

$$\pi(l_{ij}^1) = \begin{cases} \frac{9}{2}(l-j) + 6, & \text{when } i = 1, \\ \frac{9}{2}(l-j) + 7, & \text{when } i = 2, \\ \frac{9}{2}(l-j) + 8, & \text{when } i = 3, \\ \frac{9}{2}(l-j) + 9, & \text{when } i = 4, \end{cases}$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{9}{2}(l-j) + 1, & \text{when } i = 1, \\ \frac{9}{2}(l-j) + 2, & \text{when } i = 2, \\ \frac{9}{2}(l-j) + 3, & \text{when } i = 3, \\ \frac{9}{2}(l-j) + 4, & \text{when } i = 4, \\ \frac{9}{2}(l-j) + 5, & \text{when } i = 5, \end{cases}$$

Thus, the labeling π constitute the following set of edge-sum as given below;

$$\text{edge-sum} = \begin{cases} \left\{ \frac{9l+21}{2}, \frac{9l+23}{2}, \dots, \frac{27l+19}{2} \right\}, & \text{when } l \equiv 1(\text{mod } 2), \\ \left\{ \frac{9l+32}{2}, \frac{9l+34}{2}, \dots, \frac{27l+30}{2} \right\}, & \text{when } l \equiv 0(\text{mod } 2), \end{cases}$$

Therefore, by Lemma 1.2, we expand π to a super edge-magic total labeling of H □

Theorem 2.4. Prove that graph $\Gamma \cong \underbrace{(l, l, \dots, l)}_{(n-1)\text{-times}} \cup \underbrace{(l, l, \dots, l)}_{n\text{-times}}$ admits super edge-magic deficiency at most

$2(n-1)$, where $l \geq 1$.

Proof. Consider the graph $H \cong \underbrace{(l, l, \dots, l)}_{(n-1)\text{-times}} \cup \underbrace{(l, l, \dots, l)}_{n\text{-times}} \cup 2(n-1)K_1$. The vertex-set and edge-set are

defined as

$$V(H) = \{l_t : 1 \leq t \leq 2\} \cup \{l_{ij}^t : 1 \leq t \leq 2, 1 \leq i \leq t+n-2, 1 \leq j \leq l\} \cup \{z_k : 1 \leq k \leq 2(n-1)\}$$

and

$$E(H) = \{l_t l_{i1}^t : 1 \leq t \leq 2, 1 \leq i \leq t+n-2\} \cup \{l_{ij}^t l_{i(j+1)}^t : 1 \leq t \leq 2, 1 \leq i \leq t+n-2, 1 \leq j \leq l-1\}.$$

The order and size of the graph Γ are $p = 2n(l+1) - l$, and $q = (2n-1)l$. In order to show that H is super edge-magic total, we have two cases:

Case-1: When $l \equiv 1(\text{mod } 2)$

We present vertex labeling $\pi : V(H) \rightarrow \{1, 2, \dots, 2n(l+1) - l\}$ as given below:

$$\pi(l_t) = (2n-1)(l-t) + (4n-1); 1 \leq t \leq 2, \text{ and } \pi(z_k) = (2n-1)l + 1 + k; 1 \leq k \leq 2(n-1)$$

When $1 \leq j \leq l$:

For $j \equiv 1(\text{mod } 2)$;

$$\pi(l_{ij}^1) = \frac{(2n-1)}{2}(l-j) + i, \text{ when } 1 \leq i \leq n-1,$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{(2n-1)}{2}(l-j) + n + i, & \text{when } 1 \leq i \leq n-1, \\ \frac{(2n-1)}{2}(l-j) + i, & \text{when } i = n, \end{cases}$$

When $j \equiv 0(\text{mod } 2)$:

For, $1 \leq j < l-1$;

$$\pi(l_{ij}^1) = \frac{2n-1}{2}(2l-j) + 2(n-i) + 1, \text{ when } 1 \leq i \leq n-1.$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{2n-1}{2}(2l-j) + 2(n-i), & \text{when } 1 \leq i \leq n-1, \\ \frac{2n-1}{2}(2l-j) + 2(n-i) + 1, & \text{when } i = n, \end{cases}$$

For $j = l-1$;

$$\pi(l_{ij}^1) = \frac{(2n-1) + 2(2n+i) - 3}{2}, \text{ when } 1 \leq i \leq n-1$$

and

$$\pi(l_{ij}^2) = \begin{cases} \frac{(2n-1)l + 2(n+i) - 1}{2}, & \text{when } 1 \leq i \leq n-1, \\ \frac{(2n-1)l + 2(2n+i) - 3}{2}, & \text{when } i = n, \end{cases}$$

Case-2: When $l \equiv 0(\text{mod } 2)$

$$\pi(l_t) = \frac{(2n-1)(l-2t)}{2} + 4n-1; 1 \leq t \leq 2, \text{ and } \pi(z_k) = \frac{(2n-1)l}{2} + k; 1 \leq k \leq 2(n-1)$$

When $1 \leq j \leq l$;

For $j \equiv 1(\text{mod } 2)$

$$\pi(l_{ij}^1) = (2n-1)l + (i+1) - \frac{2n-1}{2}(j-1), \text{ when } 1 \leq i \leq n-1,$$

and

$$\pi(l_{ij}^2) = (2n-1)l + n + i - \frac{2n-1}{2}(j-1), \text{ when } 1 \leq i \leq n$$

When $j \equiv 0(\text{mod } 2)$;

For $1 \leq j \leq 1$;

$$\pi(l_{ij}^1) = \frac{2n-1}{2}(l-j) + n + i, \text{ when } 1 \leq i \leq n-1,$$

and

$$\pi(l_{ij}^2) = \frac{2n-1}{2}(l-j) + i, \text{ when } 1 \leq i \leq n.$$

Thus the labeling π constitute the following set of edge-sum as given below;

$$\text{edge-sum} = \begin{cases} \left\{ \frac{(2n-1)l+4n+1}{2}, \frac{(2n-1)l+4n+3}{2}, \dots, \frac{3(2n-1)l+4n-1}{2} \right\}, & \text{when } l \equiv 1 \pmod{2}, \\ \left\{ \frac{(2n-1)l+6n+2}{2}, \frac{(2n-1)l+6n+4}{2}, \dots, \frac{3(2n-1)l+6n}{2} \right\}, & \text{when } l \equiv 0 \pmod{2}, \end{cases}$$

Thus, by Lemma 1.2, we expand π to a super edge-magic total labeling of H . $\square$

3. Conclusion

In the current paper, we have proved the super edge-magic deficiency of forest for two non-isomorphic copies of subdivided stars. The problem is still for super edge-magic deficiency for n non-isomorphic copies for subdivision of stars.

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