

Novel Connection Based Zagreb Indices of Several Wheel-Related Graphs

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Abstract

The field of graph theory plays a useful role in cheminformatics especially in mathematics and chemistry to develop several chemical structures with the help of numerical parameters (or topological indices). A topological index (TI) is a function from the set of graphs to the set of real numbers that associates a real number to each molecular graph. Moreover, for two graphs K_1 and K_2 , if $K_1 \cong K_2$ then $TI(K_1) = TI(K_2)$. The first TI based on distance was introduced by Wiener (1947). Gutman and Trinajstić (1972) introduced degree and connection based TIs. The degree and connection number of a vertex $v \in K$ is the number of vertices at distance 1 and 2 respectively, from v in K . In this paper, we introduced several novel connection based Zagreb indices such as modified fourth Zagreb connection index (ZCI), multiplicative fourth ZCI, modified first multiplicative ZCI, modified second multiplicative ZCI, modified third multiplicative ZCI, connection distance index and Gutman connection index. We also computed main results in the form of exact solutions for these newly defined connection based Zagreb indices of wheel-related graphs which are wheel, gear, helm, flower and sunflower graphs.

Keywords: Connection number; Modified Zagreb indices; ; Multiplicative Zagreb indices; Modified multiplicative Zagreb indices; Wheel-related graphs.

Mathematics Subject Classification: : 05C09, 05C92

1. Introduction

Numeric number (TI) is a mathematical coding of a molecular graph which predicts the biological, toxicological, structural and physicochemical properties of a chemical compound. These are also useful for investigation of correlation values among various octane isomers, see [1, 2]. A special use of these TIs have been widely shown in the study of QSAR and QSPR [3]. In the same paper, Todeschini and Consonni showed that these relationships are abundantly used in the subject of cheminformatics. Moreover, drugs and their particles, medicinal behaviors of the drugs and pharmaceutical industries are studied by TIs, for more information [4, 5]. There are three classification of TI as degree, distance and polynomial based of the graph.

A TI based on distance is known as distance based TI. Wiener firstly introduced distance based TI in [6] (1947) when he studied on the boiling point of paraffin. Now, this descriptor is called by the name of Wiener index. The TIs and graph invariants based on distance between vertices of the graphs are widely

used in mathematical chemistry [7]. Dankelmann et al. [8] used distance based TI to compute sharp upper bounds for graphs of order n and diameter δ . Mazorodze et al. [9] used very famous distance based TI, Gutman index (or Schultz index of the second kind) to compute sharp upper bounds for graphs for every $\delta \geq 2$. They extended the results of Dankelmann et al. [8]. Das et al. [10] computed some upper and lower bounds and developed relations between degree distance and Gutman index. Xu et al. [11] established distance degree index of some derived graphs such as sub-division graph, line graph, edge-semitotal graph, vertex-semitotal graph, paraline graph and total graph.

The most famous class of TIs is degree based which depend on degree of the vertex. This class is further classified into two sub-classes degree and connection based TIs. Degree and connection based TIs were defined by Gutman and Trinajstić [1](1972) to compute the entire π -electron energy of alternant hydrocarbon by the use of well known TI named as first Zagreb index. Gutman, Rucic and Furtula defined another TIs known as second Zagreb index and third Zagreb index [12, 13]. In 2003, Nikolić et al. [14] defined modified version Zagreb indices. In 2011, Hao [15] used modified version approach and performed analysis of Zagreb and modified Zagreb indices. Dhanalakshmi et al. [16] computed several results for modified and multiplicative ZIs of operators (S(G) and R(G)). In 2013, Das et al. [17] computed upper bounds for the multiplicative Zagreb indices of operations on graphs. For more study about multiplicative and sum-multiplicative Zagreb indices, we refer to [18, 19, 20].

Recently, the concept of connection number (number of vertices which are at distance two) has been checked in mind of the reserchers. Now, they have started working on these connection based TIs rapidly. For this instance, Tang et al. [21] and Ali et al. [22] computed exact formulae for connection based indices and their modified versions of subdivision-related operations on graphs. Cao et al. [23] computed both exact and upper bounds formulae for connection based Zagreb indices of product-related graphs. Liu et al. [24] computed different results of cellular neural networks by using Zagreb connection indices. In 2021, Javaid et al. [25] computed general results for Zagreb connection index and its coindex of product-related operations on graphs. Moreover at this time, a variety of articles related to connection number (or leap degree) based indices are listed in literature, for more detail [26, 27, 28, 29, 30] references are therein.

In this paper, we compute the exact solutions for novel connection based Zagreb indices of several wheel-related graphs such as wheel, gear, helm, flower and sunflower graphs. The rest paper is decorated as follows: Section II presents the preliminary definitions of the classical/novel Zagreb indices and some special results which are used in the next sections, Section III holds the definitions of several wheel-related graphs and their main results based on novel connection based Zagreb indices, Section IV covers the main results of aforesaid graphs based on connection distance index and Gutman connection index and Section V covers conclusions.

2. Notations and Preliminaries

The vertex and edge sets are $V(K)$ and $E(K)$ for simple and connected graph K . $|V(K)|$ and $|E(K)|$ are the cardinalities of vertex set and edge set which are equal to n and e , respectively. The distance between two vertices u and v is the length of the shortest path between them. It is denoted by $d_K(u, v)$. In general [31], $N_K(v/m) = \{u \in V(K) ; d(u, v) = m\}$ is the open m -neighborhood set of v , where m represents a positive integer and $|N_K(v/m)| = d_K(v/m)$ is called m -distance degree of a vertex v . In particular,

- If $m = 1$, $d_K(v/1) = |N_K(v/1)| = d_K(v)$ = degree of v (number of vertices at distance one from v).
- If $m = 2$, $d_K(v/2) = |N_K(v/2)| = \tau_K(v)$ = connection number of v (number of vertices at distance two from v).

For more terminologies and notations, we suggest to see [23], and citations therein. $|V(K_1)| = n_1$, $|V(K_2)| = n_2$, $|E(K_1)| = e_1$ and $|E(K_2)| = e_2$ are the orders and sizes of two connected graphs K_1 and K_2

throughout the study of this paper.

Definition 2.1. For a graph K , the Wiener index ($W(K)$), degree distance index ($DD(K)$) and Gutman index ($Gut(K)$) are defined as

$$W(K) = \sum_{\{u,v\} \subseteq V(K)} d(u,v) = \frac{1}{2} \sum_{u,v \subseteq V(K)} d(u,v), \quad DD(K) = \sum_{\{u,v\} \subseteq V(K)} [d_K(u) + d_K(v)]d(u,v) \quad \text{and}$$

$$Gut(K) = \sum_{\{u,v\} \subseteq V(K)} [d_K(u) \times d_K(v)]d(u,v).$$

A TI is a numerical quantity related to a graph that is invariant under graph automorphisms. If a TI attached with the distance between vertices of any graph is called distance based TI. The Wiener index $W(K)$ is the first distance based TI. In fact, the degree distance and Gutman index are special cases of weighted Wiener index. These distance-based TIs are defined by Wiener, Dobrynin, Kochetova and Gutman, see [32, 33, 34].

Definition 2.2. For a graph K , the first Zagreb index ($M_1(K)$), second Zagreb index ($M_2(K)$) and forgotten index ($M_3(K) = F(K)$) are defined as

$$M_1(K) = \sum_{v \in V(K)} [d_K(v)]^2 = \sum_{uv \in E(K)} [d_K(u) + d_K(v)], \quad M_2(K) = \sum_{uv \in E(K)} [d_K(u) \times d_K(v)] \quad \text{and}$$

$$M_3(K) = \sum_{v \in V(K)} [d_K(v)]^3 = \sum_{uv \in V(K)} [d_K^2(u) + d_K^2(v)].$$

These degree-based TIs are defined by Gutman, Trinajstić, Ruscic and Furtula, see [1, 12, 13]. These are frequently used to predict better outcomes in molecular structures such as entropy, heat capacity, acentric factor, ZE-isomerism and absolute value of correlation coefficient [35].

Corresponding to these distance and degree based TIs, the connection based TIs are defined in Definition 2.3, 2.4 and 2.5.

Definition 2.3. For a graph K , the connection distance index ($CD(K)$) and Gutman connection index ($GC(K)$) are defined as

$$(a) \quad CD(K) = \sum_{\{u,v\} \subseteq V(K)} [\tau_K(u) + \tau_K(v)]d(u,v),$$

$$(b) \quad GC(K) = \sum_{\{u,v\} \subseteq V(K)} [\tau_K(u) \times \tau_K(v)]d(u,v).$$

Definition 2.4. For a graph K , the first Zagreb connection index ($ZC_1(K)$) and second Zagreb connection index ($ZC_2(K)$) are defined as

$$ZC_1(K) = \sum_{v \in V(K)} [\tau_K(v)]^2 \quad \text{and} \quad ZC_2(K) = \sum_{uv \in E(K)} [\tau_K(u) \times \tau_K(v)].$$

Let suppose $P_0(K)$ be the set of the isolated vertices of the graph K such that $|P_0(K)| = m_0$. The following relation was derived in [36]: $\sum_{u \in V(G) \setminus P_0(K)} d_K(u)f(d_K(u)) = \sum_{uv \in E(K)} [f(d_K(u)) + f(d_K(v))]$, where f is

defined on the set of degree of vertex set that is used as a real value function of graph K . Ali and Trinajstić [37] explained the reality of the following identity in this style: $\sum_{u \in V(K) \setminus P_0(K)} d_K(u)g(u) =$

$\sum_{uv \in E(K)} [g(u) + g(v)]$, where both the functions g and f have same properties. In this way, they redefined the

concept of Gutman and Trinajstić [1] (1972) and introduced modified first Zagreb connection index (ZC_1^*) by putting $g(u) = \tau_K(u)$ and $g(v) = \tau_K(v)$ in the above equation as $ZC_1^*(K) = \sum_{uv \in E(K)} [\tau_K(u) + \tau_K(v)]$.

In 2020, Ali et al. [22] enhanced the concept of connection based indices in the form of modified second Zagreb connection index (ZC_2^*) and modified third Zagreb connection index (ZC_3^*) by putting $g(u) = d_K(u)\tau_K(v)$ and $g(v) = d_K(v)\tau_K(u)$ or $g(u) = d_K(u)\tau_K(u)$ and $g(v) = d_K(v)\tau_K(v)$ in the above identity as

$$ZC_2^*(K) = \sum_{uv \in E(K)} [d_K(u)\tau_K(v) + d_K(v)\tau_K(u)], \quad \text{and} \quad ZC_3^*(K) = \sum_{uv \in E(K)} [d_K(u)\tau_K(u) + d_K(v)\tau_K(v)].$$

Now if we replace addition by multiplication then the extended modified Zagreb connection index is obtained as follows:

Definition 2.5. For a graph K , modified fourth Zagreb connection index ($ZC_4^*(K)$) is defined as

$$ZC_4^*(K) = \sum_{uv \in E(K)} [d_K(u)\tau_K(v) \times d_K(v)\tau_K(u)] = \sum_{uv \in E(K)} [d_K(u)\tau_K(u) \times d_K(v)\tau_K(v)].$$

Definition 2.6. For a graph K , first multiplicative Zagreb index ($MZ_1(K)$) and second multiplicative Zagreb index ($MZ_2(K)$) are defined as

$$(a) \quad MZ_1(K) = \prod_{uv \in E(K)} [d_K(u) + d_K(v)],$$

$$(b) \quad MZ_2(K) = \prod_{uv \in E(K)} [d_K(u) \times d_K(v)].$$

These degree-based TIs are defined by Ghorbani and Azimi, see [38]. Corresponding to these degree based TIs, the connection based TIs are defined in Definition 2.7 and 2.8.

Definition 2.7. For a graph K , first multiplicative Zagreb connection index ($MZC_1(K)$), second multiplicative Zagreb connection index ($MZC_2(K)$), third multiplicative Zagreb connection index ($MZC_3(K)$) and fourth multiplicative Zagreb connection index ($MZC_4(K)$) are defined as

$$(a) \quad MZC_1(K) = \prod_{v \in V(K)} [\tau_K(v)]^2,$$

$$(b) \quad MZC_2(K) = \prod_{uv \in E(K)} [\tau_K(u) \times \tau_K(v)],$$

$$(c) \quad MZC_3(K) = \prod_{v \in V(K)} [d_K(v) \times \tau_K(v)],$$

$$(d) \quad MZC_4(K) = \prod_{uv \in E(K)} [\tau_K(u) + \tau_K(v)].$$

The first three connection based multiplicative Zagreb indices was defined by Haoer et al. [39] (2020). They used the same concept in the name of multiplicative leap Zagreb indices to compute exact values of some t-thorny graphs.

Definition 2.8. For a graph K , modified first multiplicative Zagreb connection index ($MZC_1^*(K)$), modified second multiplicative Zagreb connection index ($MZC_2^*(K)$) and modified third multiplicative Zagreb connection index ($MZC_3^*(K)$) are defined as

$$(a) \quad MZC_1^*(K) = \prod_{uv \in E(K)} [d_K(u)\tau_K(v) + d_K(v)\tau_K(u)],$$

$$(b) \quad MZC_2^*(K) = \prod_{uv \in E(K)} [d_K(u)\tau_K(u) + d_K(v)\tau_K(v)],$$

$$(c) \quad MZC_3^*(K) = \prod_{uv \in E(K)} [d_K(u)\tau_K(v) \times d_K(v)\tau_K(u)] = \prod_{uv \in E(K)} [d_K(u)\tau_K(u) \times d_K(v)\tau_K(v)].$$

Now, we present some important results which are used in the main results.

Lemma 2.1.[40] Let K be a connected graph with n vertices and e edges. Then,
 $\tau_K(u) + d_K(u) \leq \sum_{v \in N_K(u)} (d_K(v))$, where equality holds if and only if K is a $\{C_3, C_4\}$ – free graph.

Lemma 2.2.[21] Let K be a connected and $\{C_3, C_4\}$ – free graph with n vertices and e edges. Then,
 (i) $\sum_{v \in V(K)} d_K(v) = 2e$, (ii) $\sum_{v \in V(K)} \tau_K(v) = M_1(K) - 2e$.

Lemma 2.3. Let W_n be a wheel graph with $n + 1$ vertices and $2n$ edges. Then,
 $\sum_{\{u,v\} \subseteq W_n / \{v_0\}} d(u, v) = n(n - 2)$.

Lemma 2.4. Let G_n be a gear graph with $2n + 1$ vertices and $3n$ edges. Then,
 (i) $\sum_{\{u,v\} \subseteq U} d(u, v) = 2n(n - 2)$, (ii) $\sum_{\{u,v\} \subseteq V} d(u, v) = n(n - 1)$, (iii) $\sum_{u \in U, v \in V} d(u, v) = n(3n - 4)$.

Lemma 2.5. Let H_n be a gear graph with $2n + 1$ vertices and $3n$ edges. Then,
 (i) $\sum_{\{u,v\} \subseteq U} d(u, v) = n(2n - 3)$, (ii) $\sum_{\{u,v\} \subseteq V} d(u, v) = n(n - 2)$, (iii) $\sum_{u \in U, v \in V} d(u, v) = n(3n - 4)$.

Lemma 2.6. Let Fl_n be a flower graph with $2n + 1$ vertices and $4n$ edges. Also, let $U = \{u_1, u_2, u_3, \dots, u_n\}$. Then, (i) $\sum_{\{u,v\} \subseteq U} d(u, v) = n(n - 1)$, (ii) $\sum_{\{u,v\} \subseteq V} d(u, v) = n(n - 2)$, (iii) $\sum_{u \in U, v \in V} d(u, v) = n(2n - 1)$.

Lemma 2.7. Let Sf_n be a sunflower graph with $3n + 1$ vertices and $5n$ edges. Also, let
 $V = \{v_1, v_2, v_3, \dots, v_n\}$ and $W = \{w_1, w_2, w_3, \dots, w_n\}$. Then,
 (i) $\sum_{\{u,v\} \subseteq U} d(u, v) = n(n - 1)$, (ii) $\sum_{\{u,v\} \subseteq V} d(u, v) = n(n - 2)$, (iii) $\sum_{u \in U, v \in V} d(u, v) = n(2n - 1)$,
 (iv) $\sum_{u \in U, v \in W} d(u, v) = 2n^2$, (v) $\sum_{u \in V, v \in W} d(u, v) = 2n^2$.

Lemma 2.8.[22] Let K be a connected graph and $K \cong P_n$, then
 (i) $ZC_2^*(K) = 8n - 22$ if $n \geq 4$, (ii) $ZC_3^*(K) = 8n - 22$ if $n \geq 3$.

3. Definitions and Main Results

This section consists of wheel-related graphs' definitions which are wheel graph, gear graph, helm graph, flower graph and sunflower graph respectively, & their main results for novel connection based Zagreb indices.

1. Wheel Graph W_n :

The wheel graph W_n is defined by joining K_1 and C_n . Where K_1 and C_n are the complete and cycle graphs of orders 1 and n respectively. So, order of wheel graph ($|V(W_n)|$) is $n + 1$ and size of wheel graph ($|E(W_n)|$) is $2n$. Apex is the vertex corresponding to K_1 and rim is the set of those vertices which is corresponding to C_n . The graphical depiction of wheel graph is presented in Figure 1.

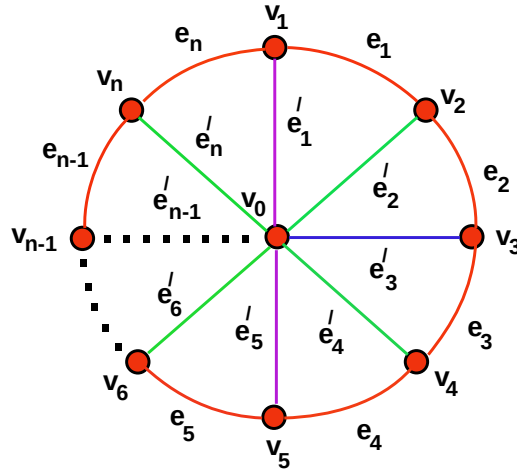


Figure 1. Wheel graph W_n

Table 1: Partition of vertices for W_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)
d_{W_n}	n	3
τ_{W_n}	0	n-3

Theorem 3.1. Let W_n be a wheel graph of order $n + 1$, then connection based Zagreb indices of W_n are

- (1) $ZC_2^*(W_n) = n(n^2 + 3n - 18)$,
- (2) $ZC_3^*(W_n) = 9n(n - 3)$,
- (3) $ZC_4^*(W_n) = 9n(n^2 - 6n + 9)$,
- (4) $MZC_1(W_n) = 0$,
- (5) $MZC_2(W_n) = 0$,
- (6) $MZC_3(W_n) = 0$,
- (7) $MZC_4(W_n) = 2^n(n - 3)^{2n}$,
- (8) $MZC_1^*(W_n) = (6n)^n(n - 3)^{2n}$,
- (9) $MZC_2^*(W_n) = (18)^n(n - 3)^{2n}$,
- (10) $MZC_3^*(W_n) = 0$.

Proof (1). Since W_n is a wheel graph of order $n + 1$, where $n \geq 4$. Also, v_0 is the apex vertex and $\{v_1, v_2, \dots, v_n\}$ is the set of rim vertices for W_n . By definition,

$$\begin{aligned}
 ZC_2^*(W_n) &= \sum_{uv \in E(W_n)} [d_{W_n}(u)\tau_{W_n}(v) + d_{W_n}(v)\tau_{W_n}(u)] \\
 &= \sum_{v_0 v_i \in E(W_n)} [d_{W_n}(v_0)\tau_{W_n}(v_i) + d_{W_n}(v_i)\tau_{W_n}(v_0)] + \sum_{v_i v_{i+1} \in E(W_n)} [d_{W_n}(v_i)\tau_{W_n}(v_{i+1}) + d_{W_n}(v_{i+1})\tau_{W_n}(v_i)] \\
 &= \sum_{i=1}^n [n(n-3) + 0] + \sum_{i=1}^n [3(n-3) + 3(n-3)] \\
 &= n(n^2 - 3n) + 2n(3n - 9)
 \end{aligned}$$

$$= n(n^2 + 3n - 18).$$

(2).

$$\begin{aligned} ZC_3^*(W_n) &= \sum_{uv \in E(W_n)} [d_{W_n}(u) \tau_{W_n}(u) + d_{W_n}(v) \tau_{W_n}(v)] \\ &= \sum_{v_0 v_i \in E(W_n)} [d_{W_n}(v_0) \tau_{W_n}(v_0) + d_{W_n}(v_i) \tau_{W_n}(v_i)] + \sum_{v_i v_{i+1} \in E(W_n)} [d_{W_n}(v_i) \tau_{W_n}(v_i) + d_{W_n}(v_{i+1}) \tau_{W_n}(v_{i+1})] \\ &= \sum_{i=1}^n [0 + 3(n-3)] + \sum_{i=1}^n [3(n-3) + 3(n-3)] \\ &= n(3n-9) + 2n(3n-9) \\ &= 9n(n-3). \end{aligned}$$

(3).

$$\begin{aligned} ZC_4^*(W_n) &= \sum_{uv \in E(W_n)} [d_{W_n}(u) \tau_{W_n}(v) \times d_{W_n}(v) \tau_{W_n}(u)] \\ &= \sum_{v_0 v_i \in E(W_n)} [d_{W_n}(v_0) \tau_{W_n}(v_i) \times d_{W_n}(v_i) \tau_{W_n}(v_0)] + \sum_{v_i v_{i+1} \in E(W_n)} [d_{W_n}(v_i) \tau_{W_n}(v_{i+1}) \times d_{W_n}(v_{i+1}) \tau_{W_n}(v_i)] \\ &= \sum_{i=1}^n [0] + \sum_{i=1}^n [3^2(n-3)^2] \\ &= 9n(n^2 - 6n + 9). \end{aligned}$$

(4).

$$\begin{aligned} MZC_1(W_n) &= \prod_{v \in V(W_n)} [\tau_{W_n}(v)]^2 \\ &= [\tau_{W_n}(v_0)]^2 \times \prod_{i=1}^n [\tau_{W_n}(v_i)]^2 \\ &= (0)^2 \times (n-3)^{2n} = 0. \end{aligned}$$

(5).

$$\begin{aligned} MZC_2(W_n) &= \prod_{uv \in E(W_n)} [\tau_{W_n}(u) \times \tau_{W_n}(v)] \\ &= \prod_{v_0 v_i \in E(W_n)} [\tau_{W_n}(v_0) \times \tau_{W_n}(v_i)] \times \prod_{v_i v_{i+1} \in E(W_n)} [\tau_{W_n}(v_i) \times \tau_{W_n}(v_{i+1})] \\ &= (0)^n \times [(n-3) \times (n-3)]^n = 0. \end{aligned}$$

(6).

$$\begin{aligned} MZC_3(W_n) &= \prod_{v \in V(W_n)} [d_{W_n}(v) \times \tau_{W_n}(v)] \\ &= [d_{W_n}(v_0) \times \tau_{W_n}(v_0)] \times \prod_{i=1}^n [d_{W_n}(v_i) \times \tau_{W_n}(v_i)] \\ &= [n \times 0] \times [3(n-3)]^n = 0. \end{aligned}$$

(7).

$$\begin{aligned}
MZC_4(W_n) &= \prod_{uv \in E(W_n)} [\tau_{W_n}(u) + \tau_{W_n}(v)] \\
&= \prod_{v_0 v_i \in E(W_n)} [\tau_{W_n}(v_0) + \tau_{W_n}(v_i)] \times \prod_{v_i v_{i+1} \in E(W_n)} [\tau_{W_n}(v_i) + \tau_{W_n}(v_{i+1})] \\
&= [0 + (n-3)]^n \times [(n-3) + (n-3)]^n \\
&= 2^n (n-3)^{2n}.
\end{aligned}$$

(8).

$$\begin{aligned}
MZC_1^*(W_n) &= \prod_{uv \in E(W_n)} [d_{W_n}(u) \tau_{W_n}(v) + d_{W_n}(v) \tau_{W_n}(u)] \\
&= \prod_{v_0 v_i \in E(W_n)} [d_{W_n}(v_0) \tau_{W_n}(v_i) + d_{W_n}(v_i) \tau_{W_n}(v_0)] \times \prod_{v_i v_{i+1} \in E(W_n)} [d_{W_n}(v_i) \tau_{W_n}(v_{i+1}) + d_{W_n}(v_{i+1}) \tau_{W_n}(v_i)] \\
&= [n(n-3) + 0]^n \times [3(n-3) + 3(n-3)]^n \\
&= n^n (n-3)^n \times 6^n (n-3)^n \\
&= (6n)^n (n-3)^{2n}.
\end{aligned}$$

(9).

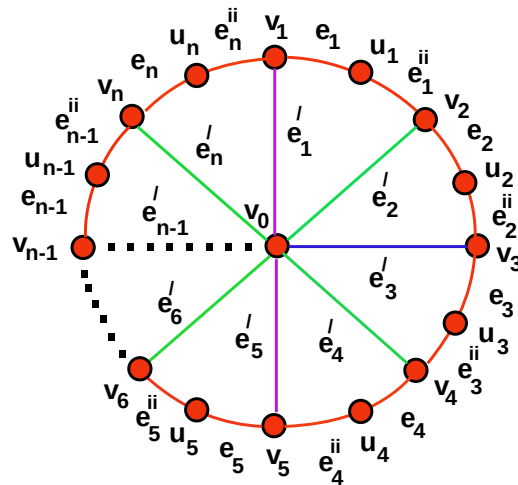
$$\begin{aligned}
MZC_2^*(W_n) &= \prod_{uv \in E(W_n)} [d_{W_n}(u) \tau_{W_n}(u) + d_{W_n}(v) \tau_{W_n}(v)] \\
&= \prod_{v_0 v_i \in E(W_n)} [d_{W_n}(v_0) \tau_{W_n}(v_0) + d_{W_n}(v_i) \tau_{W_n}(v_i)] \times \prod_{v_i v_{i+1} \in E(W_n)} [d_{W_n}(v_i) \tau_{W_n}(v_i) + d_{W_n}(v_{i+1}) \tau_{W_n}(v_{i+1})] \\
&= [0 + 3(n-3)]^n \times [3(n-3) + 3(n-3)]^n \\
&= 3^n (n-3)^n \times 6^n (n-3)^n \\
&= (18)^n (n-3)^{2n}.
\end{aligned}$$

(10).

$$\begin{aligned}
MZC_3^*(W_n) &= \prod_{uv \in E(W_n)} [d_{W_n}(u) \tau_{W_n}(v) \times d_{W_n}(v) \tau_{W_n}(u)] \\
&= \prod_{v_0 v_i \in E(W_n)} [d_{W_n}(v_0) \tau_{W_n}(v_i) \times d_{W_n}(v_i) \tau_{W_n}(v_0)] \times \prod_{v_i v_{i+1} \in E(W_n)} [d_{W_n}(v_i) \tau_{W_n}(v_{i+1}) \times d_{W_n}(v_{i+1}) \tau_{W_n}(v_i)] \\
&= (0)^n \times [3(n-3) \times 3(n-3)]^n = 0.
\end{aligned}$$

2. Gear Graph G_n :

The gear graph G_n is obtained from wheel graph W_n by including a new vertex between each pair of adjacent vertex of rim. So, order of gear graph ($|V(G_n)|$) is $2n + 1$ and size of gear graph ($|E(G_n)|$) is $3n$. Also, the bipartite wheel graph is called gear graph. The graphical depiction of gear graph is presented in Figure 2.

Figure 2. Gear graph G_n Table 2: Partition of vertices for G_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Rim (u_i)
d_{G_n}	n	3	2
τ_{G_n}	n	n-1	3

Theorem 3.2. Let G_n be a gear graph of order $2n + 1$, then novel connection based Zagreb indices of G_n are

- (1) $ZC_2^*(G_n) = n(n^2 + 6n + 14)$,
- (2) $ZC_3^*(G_n) = n(n^2 + 9n + 3)$,
- (3) $ZC_4^*(G_n) = 3n(n-1)(n^2 + 12)$,
- (4) $MZC_1(G_n) = 9^n n^2 (n-1)^{2n}$,
- (5) $MZC_2(G_n) = (9n)^n (n-1)^{3n}$,
- (6) $MZC_3(G_n) = (18)^n n^2 (n-1)^n$,
- (7) $MZC_4(G_n) = (2n-1)^n (n+2)^{2n}$,
- (8) $MZC_1^*(G_n) = (n^2 + 2n)^n (2n+7)^{2n}$,
- (9) $MZC_2^*(G_n) = 9^n (n^2 + 3n - 3)^n (n+1)^{2n}$,
- (10) $MZC_3^*(G_n) = (972)^n n^{2n} (n-1)^{3n}$.

Proof (1). Since G_n is a gear graph of order $2n + 1$. Also, v_0 is the apex vertex, $\{v_1, v_2, \dots, v_n\}$, $\{u_1, u_2, \dots, u_n\}$ are the sets of rim vertices for G_n . By definition,

$$\begin{aligned}
 ZC_2^*(G_n) &= \sum_{uv \in E(G_n)} [d_{G_n}(u)\tau_{G_n}(v) + d_{G_n}(v)\tau_{G_n}(u)] \\
 &= \sum_{v_0 v_i \in E(G_n)} [d_{G_n}(v_0)\tau_{G_n}(v_i) + d_{G_n}(v_i)\tau_{G_n}(v_0)] + \sum_{v_i u_i \in E(G_n)} [d_{G_n}(v_i)\tau_{G_n}(u_i) + d_{G_n}(u_i)\tau_{G_n}(v_i)] \\
 &= \sum_{i=1}^n [n(n-1) + 3n] + \sum_{i=1}^{2n} [(3)(3) + 2(n-1)] \\
 &= n(n^2 - n + 3n) + 2n(9 + 2n - 2)
 \end{aligned}$$

$$= n(n^2 + 6n + 14).$$

(2).

$$\begin{aligned} ZC_3^*(G_n) &= \sum_{uv \in E(G_n)} [d_{G_n}(u)\tau_{G_n}(u) + d_{G_n}(v)\tau_{G_n}(v)] \\ &= \sum_{v_0v_i \in E(G_n)} [d_{G_n}(v_0)\tau_{G_n}(v_0) + d_{G_n}(v_i)\tau_{G_n}(v_i)] + \sum_{v_iu_i \in E(G_n)} [d_{G_n}(v_i)\tau_{G_n}(v_i) + d_{G_n}(u_i)\tau_{G_n}(u_i)] \\ &= \sum_{i=1}^n [(n)(n) + 3(n-1)] + \sum_{i=1}^{2n} [(3)(n-1) + 2(3)] \\ &= n(n^2 + 9n + 3). \end{aligned}$$

(3).

$$\begin{aligned} ZC_4^*(G_n) &= \sum_{uv \in E(G_n)} [d_{G_n}(u)\tau_{G_n}(v) \times d_{G_n}(v)\tau_{G_n}(u)] \\ &= \sum_{v_0v_i \in E(G_n)} [d_{G_n}(v_0)\tau_{G_n}(v_i) \times d_{G_n}(v_i)\tau_{G_n}(v_0)] + \sum_{v_iu_i \in E(G_n)} [d_{G_n}(v_i)\tau_{G_n}(u_i) \times d_{G_n}(u_i)\tau_{G_n}(v_i)] \\ &= \sum_{i=1}^n [n(n-1) \times 3n] + \sum_{i=1}^{2n} [(3)(3) \times 2(n-1)] \\ &= 3n(n-1)(n^2 + 12). \end{aligned}$$

(4).

$$\begin{aligned} MZC_1(G_n) &= \prod_{v \in V(G_n)} [\tau_{G_n}(v)]^2 \\ &= [\tau_{G_n}(v_0)]^2 \times \prod_{i=1}^n [\tau_{G_n}(v_i)]^2 \times \prod_{i=1}^n [\tau_{G_n}(u_i)]^2 \\ &= n^2 \times (n-1)^{2n} \times 3^{2n} \\ &= 9^n n^2 (n-1)^{2n}. \end{aligned}$$

(5).

$$\begin{aligned} MZC_2(G_n) &= \prod_{uv \in E(G_n)} [\tau_{G_n}(u) \times \tau_{G_n}(v)] \\ &= \prod_{v_0v_i \in E(G_n)} [\tau_{G_n}(v_0) \times \tau_{G_n}(v_i)] \times \prod_{v_iu_i \in E(G_n)} [\tau_{G_n}(v_i) \times \tau_{G_n}(u_i)] \\ &= [n(n-1)]^n \times [(n-1)(3)]^{2n} \\ &= (9n)^n (n-1)^{3n}. \end{aligned}$$

(6).

$$\begin{aligned} MZC_3(G_n) &= \prod_{v \in V(G_n)} [d_{G_n}(v) \times \tau_{G_n}(v)] \\ &= [d_{G_n}(v_0) \times \tau_{G_n}(v_0)] \times \prod_{i=1}^n [d_{G_n}(v_i) \times \tau_{G_n}(v_i)] \times \prod_{i=1}^n [d_{G_n}(u_i) \times \tau_{G_n}(u_i)] \end{aligned}$$

$$= [(n)(n)] \times [3(n-1)]^n \times [(2)(3)]^n$$

$$= (18)^n n^2 (n-1)^n.$$

(7).

$$MZC_4(G_n) = \prod_{uv \in E(G_n)} [\tau_{G_n}(u) + \tau_{G_n}(v)]$$

$$= \prod_{v_0 v_i \in E(G_n)} [\tau_{G_n}(v_0) + \tau_{G_n}(v_i)] \times \prod_{v_i u_i \in E(G_n)} [\tau_{G_n}(v_i) + \tau_{G_n}(u_i)]$$

$$= [n+n-1]^n \times [n-1+3]^{2n}$$

$$= (2n-1)^n (n+2)^{2n}.$$

(8).

$$MZC_1^*(G_n) = \prod_{uv \in E(G_n)} [d_{G_n}(u) \tau_{G_n}(v) + d_{G_n}(v) \tau_{G_n}(u)]$$

$$= \prod_{v_0 v_i \in E(G_n)} [d_{G_n}(v_0) \tau_{G_n}(v_i) + d_{G_n}(v_i) \tau_{G_n}(v_0)] \times \prod_{v_i u_i \in E(G_n)} [d_{G_n}(v_i) \tau_{G_n}(u_i) + d_{G_n}(u_i) \tau_{G_n}(v_i)]$$

$$= [n(n-1) + 3n]^n \times [(3)(3) + 2(n-1)]^{2n}$$

$$= (n^2 + 2n)^n (2n+7)^{2n}.$$

(9).

$$MZC_2^*(G_n) = \prod_{uv \in E(G_n)} [d_{G_n}(u) \tau_{G_n}(u) + d_{G_n}(v) \tau_{G_n}(v)]$$

$$= \prod_{v_0 v_i \in E(G_n)} [d_{G_n}(v_0) \tau_{G_n}(v_0) + d_{G_n}(v_i) \tau_{G_n}(v_i)] \times \prod_{v_i u_i \in E(G_n)} [d_{G_n}(v_i) \tau_{G_n}(v_i) + d_{G_n}(u_i) \tau_{G_n}(u_i)]$$

$$= [(n)(n) + 3(n-1)]^n \times [3(n-1) + (2)(3)]^{2n}$$

$$= 9^n (n^2 + 3n - 3)^n (n+1)^{2n}.$$

(10).

$$MZC_3^*(G_n) = \prod_{uv \in E(G_n)} [d_{G_n}(u) \tau_{G_n}(v) \times d_{G_n}(v) \tau_{G_n}(u)]$$

$$= \prod_{v_0 v_i \in E(G_n)} [d_{G_n}(v_0) \tau_{G_n}(v_i) \times d_{G_n}(v_i) \tau_{G_n}(v_0)] \times \prod_{v_i u_i \in E(G_n)} [d_{G_n}(v_i) \tau_{G_n}(u_i) \times d_{G_n}(u_i) \tau_{G_n}(v_i)]$$

$$= [n(n-1) \times 3n]^n \times [(3)(3) \times 2(n-1)]^{2n}$$

$$= (972)^n n^{2n} (n-1)^{3n}.$$

3. Helm Graph H_n :

The helm graph H_n is obtained from wheel graph W_n by joining a pendant edge to the every vertex of the rim. So, order of helm graph ($|V(H_n)|$) is $2n+1$ and size of helm graph ($|E(H_n)|$) is $3n$. The graphical depiction of helm graph is presented in Figure 3.

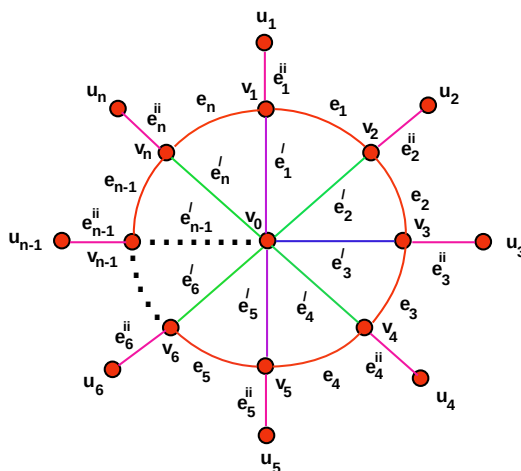


Figure 3. Helm graph H_n

Table 3: Partition of vertices for H_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Pendant (u_i)
d_{H_n}	n	4	1
τ_{H_n}	n	n-1	3

Theorem 3.3. Let H_n be a helm graph of order $2n + 1$, then novel connection based Zagreb indices of H_n are

- (1) $ZC_2^*(H_n) = n(n^2 + 12n + 3)$,
- (2) $ZC_3^*(H_n) = n(n^2 + 16n - 13)$,
- (3) $ZC_4^*(H_n) = 4n(n-1)(n^2 + 4n - 1)$,
- (4) $MZC_1(H_n) = 9^n n^2 (n-1)^{2n}$,
- (5) $MZC_2(H_n) = (3n)^n (n-1)^{4n}$,
- (6) $MZC_3(H_n) = (12)^n n^2 (n-1)^n$,
- (7) $MZC_4(H_n) = [2(n-1)(n+2)(2n-1)]^n$,
- (8) $MZC_1^*(H_n) = [8n(n+3)(n+11)(n-1)]^n$,
- (9) $MZC_2^*(H_n) = [8(n-1)(4n-1)(n^2 + 4n - 4)]^n$,
- (10) $MZC_3^*(G_n) = (768)^n n^{2n} (n-1)^{4n}$.

Proof (1). Since H_n is a helm graph of order $2n + 1$. Also, v_0 is the apex vertex. $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_n\}$ are the sets of rim and pendant vertices for H_n . By definition,

$$\begin{aligned}
 ZC_2^*(H_n) &= \sum_{uv \in E(H_n)} [d_{H_n}(u)\tau_{H_n}(v) + d_{H_n}(v)\tau_{H_n}(u)] \\
 &= \sum_{v_0 v_i \in E(H_n)} [d_{H_n}(v_0)\tau_{H_n}(v_i) + d_{H_n}(v_i)\tau_{H_n}(v_0)] + \sum_{v_i u_i \in E(H_n)} [d_{H_n}(v_i)\tau_{H_n}(u_i) + d_{H_n}(u_i)\tau_{H_n}(v_i)] \\
 &+ \sum_{v_i v_{i+1} \in E(H_n)} [d_{H_n}(v_i)\tau_{H_n}(v_{i+1}) + d_{H_n}(v_{i+1})\tau_{H_n}(v_i)]
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n [n(n-1) + 4n] + \sum_{i=1}^n [(4)(3) + 1(n-1)] + \sum_{i=1}^n [4(n-1) + 4(n-1)] \\
&= n(n^2 + 12n + 3).
\end{aligned}$$

(2).

$$\begin{aligned}
ZC_3^*(H_n) &= \sum_{uv \in E(H_n)} [d_{H_n}(u)\tau_{H_n}(u) + d_{H_n}(v)\tau_{H_n}(v)] \\
&= \sum_{v_0v_i \in E(H_n)} [d_{H_n}(v_0)\tau_{H_n}(v_0) + d_{H_n}(v_i)\tau_{H_n}(v_i)] + \sum_{v_iu_i \in E(H_n)} [d_{H_n}(v_i)\tau_{H_n}(v_i) + d_{H_n}(u_i)\tau_{H_n}(u_i)] \\
&+ \sum_{v_iv_{i+1} \in E(H_n)} [d_{H_n}(v_i)\tau_{H_n}(v_i) + d_{H_n}(v_{i+1})\tau_{H_n}(v_{i+1})] \\
&= \sum_{i=1}^n [(n)(n) + 4(n-1)] + \sum_{i=1}^n [(4)(n-1) + (1)(3)] + \sum_{i=1}^n [4(n-1) + 4(n-1)] \\
&= n(n^2 + 16n - 13).
\end{aligned}$$

(3).

$$\begin{aligned}
ZC_4^*(H_n) &= \sum_{uv \in E(H_n)} [d_{H_n}(u)\tau_{H_n}(v) \times d_{H_n}(v)\tau_{H_n}(u)] \\
&= \sum_{v_0v_i \in E(H_n)} [d_{H_n}(v_0)\tau_{H_n}(v_i) \times d_{H_n}(v_i)\tau_{H_n}(v_0)] + \sum_{v_iu_i \in E(H_n)} [d_{H_n}(v_i)\tau_{H_n}(u_i) \times d_{H_n}(u_i)\tau_{H_n}(v_i)] \\
&+ \sum_{v_iv_{i+1} \in E(H_n)} [d_{H_n}(v_i)\tau_{H_n}(v_{i+1}) \times d_{H_n}(v_{i+1})\tau_{H_n}(v_i)] \\
&= \sum_{i=1}^n [n(n-1) \times 4n] + \sum_{i=1}^n [(4)(3) \times 1(n-1)] + \sum_{i=1}^n [4(n-1) \times 4(n-1)] \\
&= 4n(n-1)(n^2 + 4n - 1).
\end{aligned}$$

(4).

$$\begin{aligned}
MZC_1(H_n) &= \prod_{v \in V(H_n)} [\tau_{H_n}(v)]^2 \\
&= [\tau_{H_n}(v_0)]^2 \times \prod_{i=1}^n [\tau_{H_n}(v_i)]^2 \times \prod_{i=1}^n [\tau_{H_n}(u_i)]^2 \\
&= n^2 \times (n-1)^{2n} \times 3^{2n} \\
&= 9^n n^2 (n-1)^{2n}.
\end{aligned}$$

(5).

$$\begin{aligned}
MZC_2(H_n) &= \prod_{uv \in E(H_n)} [\tau_{H_n}(u) \times \tau_{H_n}(v)] \\
&= \prod_{v_0v_i \in E(H_n)} [\tau_{H_n}(v_0) \times \tau_{H_n}(v_i)] \times \prod_{v_iu_i \in E(H_n)} [\tau_{H_n}(v_i) \times \tau_{H_n}(u_i)] \times \prod_{v_iv_{i+1} \in E(H_n)} [\tau_{H_n}(v_i) \times \tau_{H_n}(v_{i+1})] \\
&= [n(n-1)]^n \times [(n-1)(3)]^n \times [(n-1)(n-1)]^n. \\
&= (3n)^n (n-1)^{4n}.
\end{aligned}$$

(6).

$$\begin{aligned}
MZC_3(H_n) &= \prod_{v \in V(H_n)} [d_{H_n}(v) \times \tau_{H_n}(v)] \\
&= [d_{H_n}(v_0) \times \tau_{H_n}(v_0)] \times \prod_{i=1}^n [d_{H_n}(v_i) \times \tau_{H_n}(v_i)] \times \prod_{i=1}^n [d_{H_n}(u_i) \times \tau_{H_n}(u_i)] \\
&= [(n)(n)] \times [4(n-1)]^n \times [(1)(3)]^n \\
&= (12)^n n^2 (n-1)^n.
\end{aligned}$$

(7).

$$\begin{aligned}
MZC_4(H_n) &= \prod_{uv \in E(H_n)} [\tau_{H_n}(u) + \tau_{H_n}(v)] \\
&= \prod_{v_0 v_i \in E(H_n)} [\tau_{H_n}(v_0) + \tau_{H_n}(v_i)] \times \prod_{v_i u_i \in E(H_n)} [\tau_{H_n}(v_i) + \tau_{H_n}(u_i)] \times \prod_{v_i v_{i+1} \in E(H_n)} [\tau_{H_n}(v_i) + \tau_{H_n}(v_{i+1})] \\
&= [n + n - 1]^n \times [n - 1 + 3]^n \times [n - 1 + n - 1]^n \\
&= [2(n-1)(n+2)(2n-1)]^n.
\end{aligned}$$

(8).

$$\begin{aligned}
MZC_1^*(H_n) &= \prod_{uv \in E(H_n)} [d_{H_n}(u) \tau_{H_n}(v) + d_{H_n}(v) \tau_{H_n}(u)] \\
&= \prod_{v_0 v_i \in E(H_n)} [d_{H_n}(v_0) \tau_{H_n}(v_i) + d_{H_n}(v_i) \tau_{H_n}(v_0)] \times \prod_{v_i u_i \in E(H_n)} [d_{H_n}(v_i) \tau_{H_n}(u_i) + d_{H_n}(u_i) \tau_{H_n}(v_i)] \\
&\quad \times \prod_{v_i v_{i+1} \in E(H_n)} [d_{H_n}(v_i) \tau_{H_n}(v_{i+1}) + d_{H_n}(v_{i+1}) \tau_{H_n}(v_i)] \\
&= [n(n-1) + 4n]^n \times [(4)(3) + 1(n-1)]^n \times [4(n-1) + 4(n-1)]^n \\
&= [8n(n+3)(n+11)(n-1)]^n.
\end{aligned}$$

(9).

$$\begin{aligned}
MZC_2^*(H_n) &= \prod_{uv \in E(H_n)} [d_{H_n}(u) \tau_{H_n}(u) + d_{H_n}(v) \tau_{H_n}(v)] \\
&= \prod_{v_0 v_i \in E(H_n)} [d_{H_n}(v_0) \tau_{H_n}(v_0) + d_{H_n}(v_i) \tau_{H_n}(v_i)] \times \prod_{v_i u_i \in E(H_n)} [d_{H_n}(v_i) \tau_{H_n}(v_i) + d_{H_n}(u_i) \tau_{H_n}(u_i)] \\
&\quad \times \prod_{v_i v_{i+1} \in E(H_n)} [d_{H_n}(v_i) \tau_{H_n}(v_i) + d_{H_n}(v_{i+1}) \tau_{H_n}(v_{i+1})] \\
&= [(n)(n) + 4(n-1)]^n \times [4(n-1) + (1)(3)]^n \times [4(n-1) + (4)(n-1)]^n \\
&= [8(n-1)(4n-1)(n^2 + 4n - 4)]^n.
\end{aligned}$$

(10).

$$\begin{aligned}
MZC_3^*(H_n) &= \prod_{uv \in E(H_n)} [d_{H_n}(u) \tau_{H_n}(v) \times d_{H_n}(v) \tau_{H_n}(u)] \\
&= \prod_{v_0 v_i \in E(H_n)} [d_{H_n}(v_0) \tau_{H_n}(v_i) \times d_{H_n}(v_i) \tau_{H_n}(v_0)] \times \prod_{v_i u_i \in E(H_n)} [d_{H_n}(v_i) \tau_{H_n}(u_i) \times d_{H_n}(u_i) \tau_{H_n}(v_i)]
\end{aligned}$$

$$\begin{aligned}
 & \times \prod_{v_i v_{i+1} \in E(H_n)} [d_{H_n}(v_i) \tau_{H_n}(v_{i+1}) \times d_{H_n}(v_{i+1}) \tau_{H_n}(v_i)] \\
 & = [n(n-1) \times 4n]^n \times [(4)(3) \times 1(n-1)]^n \times [4(n-1) \times 4(n-1)]^n \\
 & = (768)^n n^{2n} (n-1)^{4n}.
 \end{aligned}$$

4. Flower Graph Fl_n :

The flower graph Fl_n is obtained from helm graph H_n by joining each pendant vertex to the apex of helm graph. So, order of flower graph ($|V(Fl_n)|$) is $2n + 1$ and size of flower graph ($|E(Fl_n)|$) is $4n$. The graphical depiction of flower graph is presented in Figure 4.

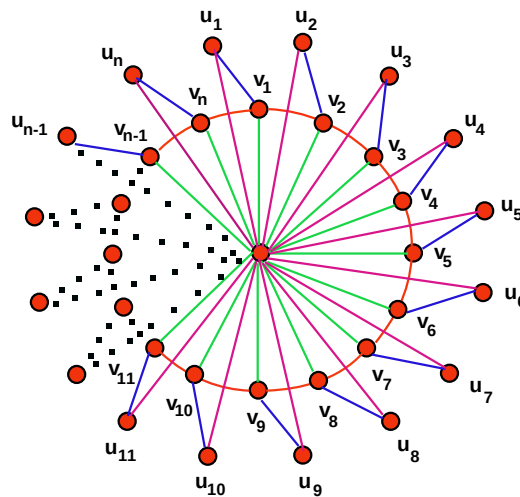


Figure 4. Flower graph Fl_n

Table 4: Partition of vertices for Fl_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Extreme (u_i)
d_{Fl_n}	$2n$	4	2
τ_{Fl_n}	0	$2n-4$	$2n-2$

Theorem 3.4. Let Fl_n be a flower graph of order $2n + 1$, then novel connection based Zagreb indices of Fl_n are

- (1) $ZC_2^*(Fl_n) = n(8n^2 + 16n - 48)$,
- (2) $ZC_3^*(Fl_n) = n(40n - 72)$,
- (3) $ZC_4^*(Fl_n) = 32n(n-2)(3n-5)$,
- (4) $MZC_1(Fl_n) = 0$,
- (5) $MZC_2(Fl_n) = 0$,
- (6) $MZC_3(Fl_n) = 0$,
- (7) $MZC_4(Fl_n) = 2^{5n}(n-2)^{2n}[(n-1)(2n-3)]^n$,
- (8) $MZC_1^*(Fl_n) = 2^{10n}[(n-1)(3n-4)]^n \times [n(n-2)]^{2n}$,
- (9) $MZC_2^*(Fl_n) = 2^{11n}(n-2)^{2n} \times [(n-1)(3n-5)]^n$,
- (10) $MZC_3^*(Fl_n) = 0$

Proof (1). Since Fl_n is a flower graph of order $2n + 1$. Also, v_0 is the apex vertex. $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_n\}$ are the sets of rim and extreme vertices for Fl_n . By definition,

$$\begin{aligned}
 ZC_2^*(Fl_n) &= \sum_{uv \in E(Fl_n)} [d_{Fl_n}(u)\tau_{Fl_n}(v) + d_{Fl_n}(v)\tau_{Fl_n}(u)] \\
 &= \sum_{v_0v_i \in E(Fl_n)} [d_{Fl_n}(v_0)\tau_{Fl_n}(v_i) + d_{Fl_n}(v_i)\tau_{Fl_n}(v_0)] + \sum_{v_0u_i \in E(Fl_n)} [d_{Fl_n}(v_0)\tau_{Fl_n}(u_i) + d_{Fl_n}(u_i)\tau_{Fl_n}(v_0)] \\
 &+ \sum_{v_iu_i \in E(Fl_n)} [d_{Fl_n}(v_i)\tau_{Fl_n}(u_i) + d_{Fl_n}(u_i)\tau_{Fl_n}(v_i)] + \sum_{v_iv_{i+1} \in E(Fl_n)} [d_{Fl_n}(v_i)\tau_{Fl_n}(v_{i+1}) + d_{Fl_n}(v_{i+1})\tau_{Fl_n}(v_i)] \\
 &= \sum_{i=1}^n [2n(2n-4) + (4)(0)] + \sum_{i=1}^n [(2n)(2n-2) + (2)(0)] + \sum_{i=1}^n [4(2n-2) + 2(2n-4)] \\
 &+ \sum_{i=1}^n [4(2n-4) + 4(2n-4)] \\
 &= n(8n^2 + 16n - 48).
 \end{aligned}$$

(2).

$$\begin{aligned}
 ZC_3^*(Fl_n) &= \sum_{uv \in E(Fl_n)} [d_{Fl_n}(u)\tau_{Fl_n}(u) + d_{Fl_n}(v)\tau_{Fl_n}(v)] \\
 &= \sum_{v_0v_i \in E(Fl_n)} [d_{Fl_n}(v_0)\tau_{Fl_n}(v_0) + d_{Fl_n}(v_i)\tau_{Fl_n}(v_i)] + \sum_{v_0u_i \in E(Fl_n)} [d_{Fl_n}(v_0)\tau_{Fl_n}(v_0) + d_{Fl_n}(u_i)\tau_{Fl_n}(u_i)] \\
 &+ \sum_{v_iu_i \in E(Fl_n)} [d_{Fl_n}(v_i)\tau_{Fl_n}(v_i) + d_{Fl_n}(u_i)\tau_{Fl_n}(u_i)] + \sum_{v_iv_{i+1} \in E(Fl_n)} [d_{Fl_n}(v_i)\tau_{Fl_n}(v_i) + d_{Fl_n}(v_{i+1})\tau_{Fl_n}(v_{i+1})] \\
 &= \sum_{i=1}^n [(2n)(0) + 4(2n-4)] + \sum_{i=1}^n [(2n)(0) + 2(2n-2)] + \sum_{i=1}^n [4(2n-4) + 2(2n-2)] \\
 &+ \sum_{i=1}^n [4(2n-4) + 4(2n-4)] \\
 &= n(40n - 72).
 \end{aligned}$$

(3).

$$\begin{aligned}
 ZC_4^*(Fl_n) &= \sum_{uv \in E(Fl_n)} [d_{Fl_n}(u)\tau_{Fl_n}(v) \times d_{Fl_n}(v)\tau_{Fl_n}(u)] \\
 &= \sum_{v_0v_i \in E(Fl_n)} [d_{Fl_n}(v_0)\tau_{Fl_n}(v_i) \times d_{Fl_n}(v_i)\tau_{Fl_n}(v_0)] + \sum_{v_0u_i \in E(Fl_n)} [d_{Fl_n}(v_0)\tau_{Fl_n}(u_i) \times d_{Fl_n}(u_i)\tau_{Fl_n}(v_0)] \\
 &+ \sum_{v_iu_i \in E(Fl_n)} [d_{Fl_n}(v_i)\tau_{Fl_n}(u_i) \times d_{Fl_n}(u_i)\tau_{Fl_n}(v_i)] + \sum_{v_iv_{i+1} \in E(Fl_n)} [d_{Fl_n}(v_i)\tau_{Fl_n}(v_{i+1}) \times d_{Fl_n}(v_{i+1})\tau_{Fl_n}(v_i)] \\
 &= \sum_{i=1}^n [2n(2n-4) \times (4)(0)] + \sum_{i=1}^n [(2n)(2n-2) \times (2)(0)] + \sum_{i=1}^n [4(2n-2) \times 2(2n-4)] \\
 &+ \sum_{i=1}^n [4(2n-4) \times 4(2n-4)] \\
 &= 32n(n-2)(3n-5).
 \end{aligned}$$

(4).

$$\begin{aligned}
 MZC_1(Fl_n) &= \prod_{v \in V(Fl_n)} [\tau_{Fl_n}(v)]^2 \\
 &= [\tau_{Fl_n}(v_0)]^2 \times \prod_{i=1}^n [\tau_{Fl_n}(v_i)]^2 \times \prod_{i=1}^n [\tau_{Fl_n}(u_i)]^2 \\
 &= (0)^2 \times (2n-4)^{2n} \times (2n-2)^{2n} \\
 &= 0.
 \end{aligned}$$

(5).

$$\begin{aligned}
 MZC_2(Fl_n) &= \prod_{uv \in E(Fl_n)} [\tau_{Fl_n}(u) \times \tau_{Fl_n}(v)] \\
 &= \prod_{v_0 v_i \in E(Fl_n)} [\tau_{Fl_n}(v_0) \times \tau_{Fl_n}(v_i)] \times \prod_{v_0 u_i \in E(Fl_n)} [\tau_{Fl_n}(v_0) \times \tau_{Fl_n}(u_i)] \times \prod_{v_i u_i \in E(Fl_n)} [\tau_{Fl_n}(v_i) \times \tau_{Fl_n}(u_i)] \\
 &\times \prod_{v_i v_{i+1} \in E(Fl_n)} [\tau_{Fl_n}(v_i) \times \tau_{Fl_n}(v_{i+1})] \\
 &= [(0)(2n-4)]^n \times [(0)(2n-2)]^n \times [(2n-4)(2n-2)]^n \times [(2n-4)(2n-4)]^n. \\
 &= 0.
 \end{aligned}$$

(6).

$$\begin{aligned}
 MZC_3(Fl_n) &= \prod_{v \in V(Fl_n)} [d_{Fl_n}(v) \times \tau_{Fl_n}(v)] \\
 &= [d_{Fl_n}(v_0) \times \tau_{Fl_n}(v_0)] \times \prod_{i=1}^n [d_{Fl_n}(v_i) \times \tau_{Fl_n}(v_i)] \times \prod_{i=1}^n [d_{Fl_n}(u_i) \times \tau_{Fl_n}(u_i)] \\
 &= [(2n)(0)] \times [4(2n-4)]^n \times [2(2n-2)]^n \\
 &= 0.
 \end{aligned}$$

(7).

$$\begin{aligned}
 MZC_4(Fl_n) &= \prod_{uv \in E(Fl_n)} [\tau_{Fl_n}(u) + \tau_{Fl_n}(v)] \\
 &= \prod_{v_0 v_i \in E(Fl_n)} [\tau_{Fl_n}(v_0) + \tau_{Fl_n}(v_i)] \times \prod_{v_0 u_i \in E(Fl_n)} [\tau_{Fl_n}(v_0) + \tau_{Fl_n}(u_i)] \times \prod_{v_i u_i \in E(Fl_n)} [\tau_{Fl_n}(v_i) + \tau_{Fl_n}(u_i)] \\
 &\times \prod_{v_i v_{i+1} \in E(Fl_n)} [\tau_{Fl_n}(v_i) + \tau_{Fl_n}(v_{i+1})] \\
 &= [0 + 2n - 4]^n \times [0 + 2n - 2]^n \times [2n - 4 + 2n - 2]^n \times [2n - 4 + 2n - 4]^n \\
 &= 2^{5n} (n-2)^{2n} [(n-1)(2n-3)]^n.
 \end{aligned}$$

(8).

$$\begin{aligned}
 MZC_1^*(Fl_n) &= \prod_{uv \in E(Fl_n)} [d_{Fl_n}(u) \tau_{Fl_n}(v) + d_{Fl_n}(v) \tau_{Fl_n}(u)] \\
 &= \prod_{v_0 v_i \in E(Fl_n)} [d_{Fl_n}(v_0) \tau_{Fl_n}(v_i) + d_{Fl_n}(v_i) \tau_{Fl_n}(v_0)] \times \prod_{v_0 u_i \in E(Fl_n)} [d_{Fl_n}(v_0) \tau_{Fl_n}(u_i) + d_{Fl_n}(u_i) \tau_{Fl_n}(v_0)]
 \end{aligned}$$

$$\begin{aligned}
& \times \prod_{v_i u_i \in E(Fl_n)} [d_{Fl_n}(v_i) \tau_v(u_i) + d_{Fl_n}(u_i) \tau_{Fl_n}(v_i)] \times \prod_{v_i v_{i+1} \in E(Fl_n)} [d_{Fl_n}(v_i) \tau_v(v_{i+1}) + d_{Fl_n}(v_{i+1}) \tau_{Fl_n}(v_i)] \\
& = [2n(2n-4) + (4)(0)]^n \times [(2n)(2n-2) + (2)(0)]^n \times [4(2n-2) + 2(2n-4)]^n \times [4(2n-4) + 4(2n-4)]^n \\
& = 2^{10n} [(n-1)(3n-4)]^n \times [n(n-2)]^{2n}.
\end{aligned}$$

(9).

$$\begin{aligned}
MZC_2^*(Fl_n) &= \prod_{uv \in E(Fl_n)} [d_{Fl_n}(u) \tau_{Fl_n}(u) + d_{Fl_n}(v) \tau_{Fl_n}(v)] \\
&= \prod_{v_0 v_i \in E(Fl_n)} [d_{Fl_n}(v_0) \tau_{Fl_n}(v_0) + d_{Fl_n}(v_i) \tau_{Fl_n}(v_i)] \times \prod_{v_0 u_i \in E(Fl_n)} [d_{Fl_n}(v_0) \tau_{Fl_n}(v_0) + d_{Fl_n}(u_i) \tau_{Fl_n}(u_i)] \\
&\times \prod_{v_i u_i \in E(Fl_n)} [d_{Fl_n}(v_i) \tau_{Fl_n}(v_i) + d_{Fl_n}(u_i) \tau_{Fl_n}(u_i)] \times \prod_{v_i v_{i+1} \in E(Fl_n)} [d_{Fl_n}(v_i) \tau_{Fl_n}(v_i) + d_{Fl_n}(v_{i+1}) \tau_{Fl_n}(v_{i+1})] \\
&= [(2n)(0) + 4(2n-4)]^n \times [(2n)(0) + (2)(2n-2)]^n \times [4(2n-4) + 2(2n-2)]^n \times [4(2n-4) + 4(2n-4)]^n \\
&= 2^{11n} (n-2)^{2n} \times [(n-1)(3n-5)]^n.
\end{aligned}$$

(10).

$$\begin{aligned}
MZC_3^*(Fl_n) &= \prod_{uv \in E(Fl_n)} [d_{Fl_n}(u) \tau_{Fl_n}(v) \times d_{Fl_n}(v) \tau_{Fl_n}(u)] \\
&= \prod_{v_0 v_i \in E(Fl_n)} [d_{Fl_n}(v_0) \tau_{Fl_n}(v_i) \times d_{Fl_n}(v_i) \tau_{Fl_n}(v_0)] \times \prod_{v_0 u_i \in E(Fl_n)} [d_{Fl_n}(v_0) \tau_{Fl_n}(u_i) \times d_{Fl_n}(u_i) \tau_{Fl_n}(v_0)] \\
&\times \prod_{v_i u_i \in E(Fl_n)} [d_{Fl_n}(v_i) \tau_{Fl_n}(u_i) \times d_{Fl_n}(u_i) \tau_{Fl_n}(v_i)] \times \prod_{v_i v_{i+1} \in E(Fl_n)} [d_{Fl_n}(v_i) \tau_v(v_{i+1}) \times d_{Fl_n}(v_{i+1}) \tau_{Fl_n}(v_i)] \\
&= [(2n)(2n-4) \times (4)(0)]^n \times [(2n)(2n-2) \times (2)(0)]^n \times [4(2n-2) \times 2(2n-4)]^n \times [4(2n-4) \times 4(2n-4)]^n \\
&= 0.
\end{aligned}$$

5. Sunflower Graph Sf_n :

The sunflower graph Sf_n is obtained from flower graph Fl_n by growing n pendant edges to the apex of flower graph. So, order of sunflower graph ($|V(Sf_n)|$) is $3n + 1$ and size of sunflower graph ($|E(Sf_n)|$) is $5n$. The graphical depiction of sunflower graph is presented in Figure 5.

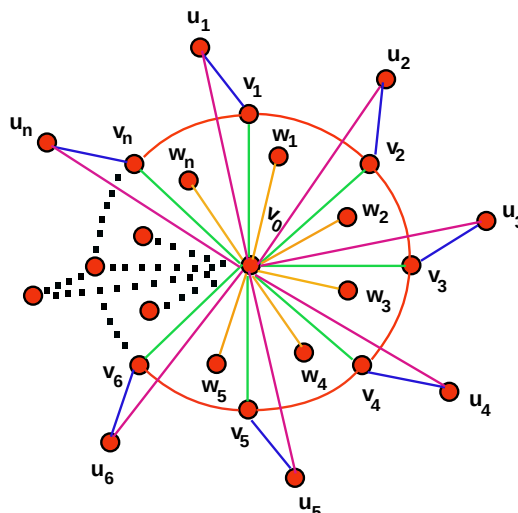


Table 5: Partition of vertices for Sf_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Extreme (u_i)	Pendant (w_i)
d_{Sf_n}	$3n$	4	2	1
τ_{Sf_n}	0	$3n-4$	$3n-2$	$3n-1$

Figure 5. Sunflower graph Sf_n

Theorem 3.5. Let Sf_n be a sunflower graph of order $3n + 1$, then novel connection based Zagreb indices of Sf_n are

- (1) $ZC_2^*(Sf_n) = (3n)(9n + 16)(n - 1)$,
- (2) $ZC_3^*(Sf_n) = n(63n - 73)$,
- (3) $ZC_4^*(Sf_n) = (8n)(3n - 4)(9n - 10)$,
- (4) $MZC_1(Sf_n) = 0$,
- (5) $MZC_2(Sf_n) = 0$,
- (6) $MZC_3(Sf_n) = 0$,
- (7) $MZC_4(Sf_n) = (3n - 4)^{2n} \times [(12)(n - 1)(3n - 1)(3n - 2)]^n$,
- (8) $MZC_1^*(Sf_n) = n^{3n} \times (3n - 4)^{2n} \times [(432)(3n - 1)(3n - 2)(9n - 8)]^n$,
- (9) $MZC_2^*(Sf_n) = (3n - 4)^{2n} \times [(128)(3n - 1)(3n - 2)(9n - 10)]^n$,
- (10) $MZC_3^*(Sf_n) = 0$.

Proof (1). Since Sf_n is a sunflower graph of order $3n + 1$. Also, v_0 is the apex vertex. $\{v_1, v_2, \dots, v_n\}$, $\{u_1, u_2, \dots, u_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are the sets of rim, extreme and pendant vertices for Sf_n . By definition,

$$\begin{aligned}
 ZC_2^*(Sf_n) &= \sum_{uv \in E(Sf_n)} [d_{Sf_n}(u)\tau_{Sf_n}(v) + d_{Sf_n}(v)\tau_{Sf_n}(u)] \\
 &= \sum_{v_0v_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(v_i) + d_{Sf_n}(v_i)\tau_{Sf_n}(v_0)] + \sum_{v_0u_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(u_i) + d_{Sf_n}(u_i)\tau_{Sf_n}(v_0)] \\
 &+ \sum_{v_0w_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(w_i) + d_{Sf_n}(w_i)\tau_{Sf_n}(v_0)] + \sum_{v_iu_i \in E(Sf_n)} [d_{Sf_n}(v_i)\tau_{Sf_n}(u_i) + d_{Sf_n}(u_i)\tau_{Sf_n}(v_i)] \\
 &+ \sum_{v_iv_{i+1} \in E(Sf_n)} [d_{Sf_n}(v_i)\tau_{Sf_n}(v_{i+1}) + d_{Sf_n}(v_{i+1})\tau_{Sf_n}(v_i)] \\
 &= \sum_{i=1}^n [(3n)(3n - 4) + (4)(0)] + \sum_{i=1}^n [(3n)(3n - 2) + (2)(0)] + \sum_{i=1}^n [(3n)(3n - 1) + (1)(0)] \\
 &+ \sum_{i=1}^n [4(3n - 2) + 2(3n - 4)] + \sum_{i=1}^n [4(3n - 4) + 4(3n - 4)] \\
 &= 3n(9n + 16)(n - 1).
 \end{aligned}$$

(2).

$$ZC_3^*(Sf_n) = \sum_{uv \in E(Sf_n)} [d_{Sf_n}(u)\tau_{Sf_n}(u) + d_{Sf_n}(v)\tau_{Sf_n}(v)]$$

$$\begin{aligned}
&= \sum_{v_0 v_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(v_0) + d_{Sf_n}(v_i) \tau_{Sf_n}(v_i)] + \sum_{v_0 u_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(v_0) + d_{Sf_n}(u_i) \tau_{Sf_n}(u_i)] \\
&+ \sum_{v_0 w_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(v_0) + d_{Sf_n}(w_i) \tau_{Sf_n}(w_i)] + \sum_{v_i u_i \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(v_i) + d_{Sf_n}(u_i) \tau_{Sf_n}(u_i)] \\
&+ \sum_{v_i v_{i+1} \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(v_i) + d_{Sf_n}(v_{i+1}) \tau_{Sf_n}(v_{i+1})] \\
&= \sum_{i=1}^n [(3n)(0) + 4(3n-4)] + \sum_{i=1}^n [(3n)(0) + 2(3n-2)] + \sum_{i=1}^n [(3n)(0) + 1(3n-1)] \\
&+ \sum_{i=1}^n [4(3n-4) + 2(3n-2)] + \sum_{i=1}^n [4(3n-4) + 4(3n-4)] \\
&= n(63n-73).
\end{aligned}$$

(3).

$$\begin{aligned}
ZC_4^*(Sf_n) &= \sum_{uv \in E(Sf_n)} [d_{Sf_n}(u) \tau_{Sf_n}(v) \times d_{Sf_n}(v) \tau_{Sf_n}(u)] \\
&= \sum_{v_0 v_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(v_i) \times d_{Sf_n}(v_i) \tau_{Sf_n}(v_0)] + \sum_{v_0 u_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(u_i) \times d_{Sf_n}(u_i) \tau_{Sf_n}(v_0)] \\
&+ \sum_{v_0 w_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(w_i) \times d_{Sf_n}(w_i) \tau_{Sf_n}(v_0)] + \sum_{v_i u_i \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(u_i) \times d_{Sf_n}(u_i) \tau_{Sf_n}(v_i)] \\
&+ \sum_{v_i v_{i+1} \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(v_{i+1}) \times d_{Sf_n}(v_{i+1}) \tau_{Sf_n}(v_i)] \\
&= \sum_{i=1}^n [3n(3n-4) \times (4)(0)] + \sum_{i=1}^n [(3n)(3n-2) \times (2)(0)] + \sum_{i=1}^n [(3n)(3n-1) \times (1)(0)] \\
&+ \sum_{i=1}^n [4(3n-2) \times 2(3n-4)] + \sum_{i=1}^n [4(3n-4) \times 4(3n-4)] \\
&= 8n(3n-4)(9n-10).
\end{aligned}$$

(4).

$$\begin{aligned}
MZC_1(Sf_n) &= \prod_{v \in V(Sf_n)} [\tau_{Sf_n}(v)]^2 \\
&= [\tau_{Sf_n}(v_0)]^2 \times \prod_{i=1}^n [\tau_{Sf_n}(v_i)]^2 \times \prod_{i=1}^n [\tau_{Sf_n}(u_i)]^2 \times \prod_{i=1}^n [\tau_{Sf_n}(w_i)]^2 \\
&= (0)^2 \times (3n-4)^{2n} \times (3n-2)^{2n} \times (3n-1)^{2n} \\
&= 0.
\end{aligned}$$

(5).

$$\begin{aligned}
MZC_2(Sf_n) &= \prod_{uv \in E(Sf_n)} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(v)] \\
&= \prod_{v_0 v_i \in E(Sf_n)} [\tau_{Sf_n}(v_0) \times \tau_{Sf_n}(v_i)] \times \prod_{v_0 u_i \in E(Sf_n)} [\tau_{Sf_n}(v_0) \times \tau_{Sf_n}(u_i)] \times \prod_{v_0 w_i \in E(Sf_n)} [\tau_{Sf_n}(v_0) \times \tau_{Sf_n}(w_i)] \\
&\times \prod_{v_i u_i \in E(Sf_n)} [\tau_{Sf_n}(v_i) \times \tau_{Sf_n}(u_i)] \times \prod_{v_i v_{i+1} \in E(Sf_n)} [\tau_{Sf_n}(v_i) \times \tau_{Sf_n}(v_{i+1})]
\end{aligned}$$

$$= [(0)(3n-4)]^n \times [(0)(3n-2)]^n \times [(0)(3n-1)]^n \times [(3n-4)(3n-2)]^n \times [(3n-4)(3n-4)]^n \\ = 0.$$

(6).

$$MZC_3(Sf_n) = \prod_{v \in V(Sf_n)} [d_{Sf_n}(v) \times \tau_{Sf_n}(v)] \\ = [d_{Sf_n}(v_0) \times \tau_{Sf_n}(v_0)] \times \prod_{i=1}^n [d_{Sf_n}(v_i) \times \tau_{Sf_n}(v_i)] \times \prod_{i=1}^n [d_{Sf_n}(u_i) \times \tau_{Sf_n}(u_i)] \times \prod_{i=1}^n [d_{Sf_n}(w_i) \times \tau_{Sf_n}(w_i)] \\ = [(3n)(0)] \times [4(3n-4)]^n \times [(2(3n-2))]^n \times [(1(3n-1))]^n \\ = 0.$$

(7).

$$MZC_4(Sf_n) = \prod_{uv \in E(Sf_n)} [\tau_{Sf_n}(u) + \tau_{Sf_n}(v)] \\ = \prod_{v_0v_i \in E(Sf_n)} [\tau_{Sf_n}(v_0) + \tau_{Sf_n}(v_i)] \times \prod_{v_0u_i \in E(Sf_n)} [\tau_{Sf_n}(v_0) + \tau_{Sf_n}(u_i)] \times \prod_{v_0w_i \in E(Sf_n)} [\tau_{Sf_n}(v_0) + \tau_{Sf_n}(w_i)] \\ \times \prod_{v_iu_i \in E(Sf_n)} [\tau_{Sf_n}(v_i) + \tau_{Sf_n}(u_i)] \times \prod_{v_iv_{i+1} \in E(Sf_n)} [\tau_{Sf_n}(v_i) + \tau_{Sf_n}(v_{i+1})] \\ = [0+3n-4]^n \times [0+3n-2]^n \times [0+3n-1]^n \times [3n-4+3n-2]^n \times [3n-4+3n-4]^n \\ = (3n-4)^{2n} \times [12(n-1)(3n-1)(3n-2)]^n.$$

(8).

$$MZC_1^*(Sf_n) = \prod_{uv \in E(Sf_n)} [d_{Sf_n}(u)\tau_{Sf_n}(v) + d_{Sf_n}(v)\tau_{Sf_n}(u)] \\ = \prod_{v_0v_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(v_i) + d_{Sf_n}(v_i)\tau_{Sf_n}(v_0)] \times \prod_{v_0u_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(u_i) + d_{Sf_n}(u_i)\tau_{Sf_n}(v_0)] \\ \times \prod_{v_0w_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(w_i) + d_{Sf_n}(w_i)\tau_{Sf_n}(v_0)] \times \prod_{v_iu_i \in E(Sf_n)} [d_{Sf_n}(v_i)\tau_{Sf_n}(u_i) + d_{Sf_n}(u_i)\tau_{Sf_n}(v_i)] \\ \times \prod_{v_iv_{i+1} \in E(Sf_n)} [d_{Sf_n}(v_i)\tau_{Sf_n}(v_{i+1}) + d_{Sf_n}(v_{i+1})\tau_{Sf_n}(v_i)] \\ = [(3n)(3n-4) + (4)(0)]^n \times [(3n)(3n-2) + (2)(0)]^n \times [(3n)(3n-1) + (1)(0)]^n \times [4(3n-2) + 2(3n-4)]^n \\ \times [4(3n-4) + 4(3n-4)]^n \\ = n^{3n} \times (3n-4)^{2n} \times [(432)(3n-1)(3n-2)(9n-8)]^n.$$

(9).

$$MZC_2^*(Sf_n) = \prod_{uv \in E(Sf_n)} [d_{Sf_n}(u)\tau_{Sf_n}(u) + d_{Sf_n}(v)\tau_{Sf_n}(v)] \\ = \prod_{v_0v_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(v_0) + d_{Sf_n}(v_i)\tau_{Sf_n}(v_i)] \times \prod_{v_0u_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(v_0) + d_{Sf_n}(u_i)\tau_{Sf_n}(u_i)] \\ \times \prod_{v_0w_i \in E(Sf_n)} [d_{Sf_n}(v_0)\tau_{Sf_n}(v_0) + d_{Sf_n}(w_i)\tau_{Sf_n}(w_i)] \times \prod_{v_iu_i \in E(Sf_n)} [d_{Sf_n}(v_i)\tau_{Sf_n}(v_i) + d_{Sf_n}(u_i)\tau_{Sf_n}(u_i)]$$

$$\begin{aligned}
& \times \prod_{v_i v_{i+1} \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(v_i) + d_{Sf_n}(v_{i+1}) \tau_{Sf_n}(v_{i+1})] \\
& = [(3n)(0) + 4(3n-4)]^n \times [(3n)(0) + (2)(3n-2)]^n \times [(3n)(0) + 1(3n-1)]^n \times [4(3n-4) + 2(3n-2)]^n \\
& \times [4(3n-4) + 4(3n-4)]^n \\
& = (3n-4)^{2n} \times [(128)(3n-1)(3n-2)(9n-10)]^n.
\end{aligned}$$

(10).

$$\begin{aligned}
MZC_3^*(Sf_n) &= \prod_{uv \in E(Sf_n)} [d_{Sf_n}(u) \tau_{Sf_n}(v) \times d_{Sf_n}(v) \tau_{Sf_n}(u)] \\
&= \prod_{v_0 v_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(v_i) \times d_{Sf_n}(v_i) \tau_{Sf_n}(v_0)] \times \prod_{v_0 u_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(u_i) \times d_{Sf_n}(u_i) \tau_{Sf_n}(v_0)] \\
&\times \prod_{v_0 w_i \in E(Sf_n)} [d_{Sf_n}(v_0) \tau_{Sf_n}(w_i) \times d_{Sf_n}(w_i) \tau_{Sf_n}(v_0)] \times \prod_{v_i u_i \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(u_i) \times d_{Sf_n}(u_i) \tau_{Sf_n}(v_i)] \\
&\times \prod_{v_i v_{i+1} \in E(Sf_n)} [d_{Sf_n}(v_i) \tau_{Sf_n}(v_{i+1}) \times d_{Sf_n}(v_{i+1}) \tau_{Sf_n}(v_i)] \\
&= [(3n)(3n-4) \times (4)(0)]^n \times [(3n)(3n-2) \times (2)(0)]^n \times [(3n)(3n-1) \times (1)(0)]^n \times [4(3n-2) \times 2(3n-4)]^n \\
&\times [4(3n-4) \times 2(3n-4)]^n \\
&= 0.
\end{aligned}$$

4. Main Results

This section consists of connection distance index and Gutman connection index for wheel-related graphs which are wheel graph, gear graph, helm graph, flower graph and sunflower graph respectively.

Theorem 4.1. Let W_n be a wheel graph of order $n+1$, where $n \geq 4$. Then, connection distance index and Gutman connection index for W_n are

$$(1) \quad CD(W_n) = n(n-3)(2n-3),$$

$$(2) \quad GC(W_n) = n(n-2)(n-3)^2.$$

Proof (1). Since W_n is a wheel graph of order $n+1$. By definition,

$$\begin{aligned}
CD(W_n) &= \sum_{\{u,v\} \subseteq W_n} [\tau_{W_n}(u) + \tau_{W_n}(v)] d(u, v) \\
&= \sum_{i=1}^n [\tau_{W_n}(v_0) + \tau_{W_n}(v_i)] d(v_0, v_i) + \sum_{\{u,v\} \subseteq W_n / \{v_0\}} [\tau_{W_n}(u) + \tau_{W_n}(v)] d(u, v) \\
&= \sum_{i=1}^n (0+n-3)1 + \sum_{\{u,v\} \subseteq W_n / \{v_0\}} (n-3+n-3) d(u, v) \\
&= n(n-3) + 2(n-3) \sum_{\{u,v\} \subseteq W_n / \{v_0\}} d(u, v)
\end{aligned}$$

Using Lemma 2.3, we have

$$= n(n-3)(2n-3).$$

(2).

$$\begin{aligned}
GC(W_n) &= \sum_{\{u,v\} \subseteq W_n} [\tau_{W_n}(u) \times \tau_{W_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{W_n}(v_0) \times \tau_{W_n}(v_i)]d(v_0, v_i) + \sum_{\{u,v\} \subseteq W_n/\{v_0\}} [\tau_{W_n}(u) \times \tau_{W_n}(v)]d(u,v) \\
&= \sum_{i=1}^n (0)(n-3)1 + \sum_{\{u,v\} \subseteq W_n/\{v_0\}} (n-3)(n-3)d(u,v) \\
&= (n-3)^2 \sum_{\{u,v\} \subseteq W_n/\{v_0\}} d(u,v) \\
&= n(n-2)(n-3)^2.
\end{aligned}$$

Theorem 4.2. Let G_n be a wheel graph of order $2n+1$. Then, connection distance index and Gutman connection index for G_n are

$$(1) \quad CD(G_n) = n(5n^2 + 12n - 23),$$

$$(2) \quad GC(G_n) = n(n^3 + 7n^2 + 5n - 7).$$

Proof (1). Since G_n is a gear graph of order $2n+1$. By definition,

$$\begin{aligned}
CD(G_n) &= \sum_{\{u,v\} \subseteq G_n} [\tau_{G_n}(u) + \tau_{G_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{G_n}(v_0) + \tau_{G_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{G_n}(v_0) + \tau_{G_n}(u_i)]d(v_0, u_i) + \sum_{\{u,v\} \subseteq U} [\tau_{G_n}(u) + \tau_{G_n}(v)]d(u,v) \\
&+ \sum_{\{u,v\} \subseteq V} [\tau_{G_n}(u) + \tau_{G_n}(v)]d(u,v) + \sum_{u \in U, v \in V} [\tau_{G_n}(u) + \tau_{G_n}(v)]d(u,v) \\
&= \sum_{i=1}^n (n+n-1)1 + \sum_{i=1}^n (n+3)2 + \sum_{\{u,v\} \subseteq U} (3+3)d(u,v) + \sum_{\{u,v\} \subseteq V} (n-1+n-1)d(u,v) \\
&+ \sum_{u \in U, v \in V} (3+n-1)d(u,v) \\
&= n(2n-1) + 2n(n+3) + 6 \sum_{\{u,v\} \subseteq U} d(u,v) + 2(n-1) \sum_{\{u,v\} \subseteq V} d(u,v) + (n+2) \sum_{u \in U, v \in V} d(u,v)
\end{aligned}$$

Using Lemma 2.4, we have

$$= n(5n^2 + 12n - 23).$$

(2).

$$\begin{aligned}
GC(G_n) &= \sum_{\{u,v\} \subseteq G_n} [\tau_{G_n}(u) \times \tau_{G_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{G_n}(v_0) \times \tau_{G_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{G_n}(v_0) \times \tau_{G_n}(u_i)]d(v_0, u_i) + \sum_{\{u,v\} \subseteq U} [\tau_{G_n}(u) \times \tau_{G_n}(v)]d(u,v) \\
&+ \sum_{\{u,v\} \subseteq V} [\tau_{G_n}(u) \times \tau_{G_n}(v)]d(u,v) + \sum_{u \in U, v \in V} [\tau_{G_n}(u) \times \tau_{G_n}(v)]d(u,v)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n (n)(n-1)(1) + \sum_{i=1}^n (3n)(2) + \sum_{\{u,v\} \subseteq U} (3)(3)d(u,v) + \sum_{\{u,v\} \subseteq V} (n-1)(n-1)d(u,v) \\
&+ \sum_{u \in U, v \in V} 3(n-1)d(u,v) \\
&= n^2(n-1) + 6n^2 + 9 \sum_{\{u,v\} \subseteq U} d(u,v) + (n-1)^2 \sum_{\{u,v\} \subseteq V} d(u,v) + 3(n-1) \sum_{u \in U, v \in V} d(u,v) \\
&= n(n^3 + 7n^2 + 5n - 7).
\end{aligned}$$

Theorem 4.3. Let H_n be a helm graph of order $2n + 1$. Then, connection distance index and Gutman connection index for H_n are

$$(1) \quad CD(H_n) = n(n-1)(5n+17),$$

$$(2) \quad GC(H_n) = n(n^3 + 6n^2 + 6n - 16).$$

Proof (1). Since H_n is a helm graph of order $2n + 1$. By definition,

$$\begin{aligned}
CD(H_n) &= \sum_{\{u,v\} \subseteq H_n} [\tau_{H_n}(u) + \tau_{H_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{H_n}(v_0) + \tau_{H_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{H_n}(v_0) + \tau_{H_n}(u_i)]d(v_0, u_i) + \sum_{\{u,v\} \subseteq U} [\tau_{H_n}(u) + \tau_{H_n}(v)]d(u,v) \\
&+ \sum_{\{u,v\} \subseteq V} [\tau_{H_n}(u) + \tau_{H_n}(v)]d(u,v) + \sum_{u \in U, v \in V} [\tau_{H_n}(u) + \tau_{H_n}(v)]d(u,v) \\
&= \sum_{i=1}^n (n+n-1)1 + \sum_{i=1}^n (n+3)2 + \sum_{\{u,v\} \subseteq U} (3+3)d(u,v) + \sum_{\{u,v\} \subseteq V} (n-1+n-1)d(u,v) \\
&+ \sum_{u \in U, v \in V} (3+n-1)d(u,v) \\
&= n(2n-1) + 2n(n+3) + 6 \sum_{\{u,v\} \subseteq U} d(u,v) + 2(n-1) \sum_{\{u,v\} \subseteq V} d(u,v) + (n+2) \sum_{u \in U, v \in V} d(u,v)
\end{aligned}$$

Using Lemma 2.5, we have

$$= n(n-1)(5n+17).$$

(2).

$$\begin{aligned}
GC(H_n) &= \sum_{\{u,v\} \subseteq H_n} [\tau_{H_n}(u) \times \tau_{H_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{H_n}(v_0) \times \tau_{H_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{H_n}(v_0) \times \tau_{H_n}(u_i)]d(v_0, u_i) + \sum_{\{u,v\} \subseteq U} [\tau_{H_n}(u) \times \tau_{H_n}(v)]d(u,v) \\
&+ \sum_{\{u,v\} \subseteq V} [\tau_{H_n}(u) \times \tau_{H_n}(v)]d(u,v) + \sum_{u \in U, v \in V} [\tau_{H_n}(u) \times \tau_{H_n}(v)]d(u,v) \\
&= \sum_{i=1}^n (n)(n-1)(1) + \sum_{i=1}^n (3n)(2) + \sum_{\{u,v\} \subseteq U} (3)(3)d(u,v) + \sum_{\{u,v\} \subseteq V} (n-1)(n-1)d(u,v) \\
&+ \sum_{u \in U, v \in V} 3(n-1)d(u,v)
\end{aligned}$$

$$\begin{aligned}
&= n(n-1)^2 + 6n^2 + 9 \sum_{\{u,v\} \subseteq U} d(u,v) + (n-1)^2 \sum_{\{u,v\} \subseteq V} d(u,v) + 3(n-1) \sum_{u \in U, v \in V} d(u,v) \\
&= n(n^3 + 6n^2 + 6n - 16).
\end{aligned}$$

Theorem 4.4. Let Fl_n be a flower graph of order $2n + 1$. Then, connection distance index and Gutman connection index for Fl_n are

$$(1) \quad CD(Fl_n) = 2n(4n^2 - 17n + 12),$$

$$(2) \quad GC(Fl_n) = 4n(n^3 - 8n^2 - 20n - 16).$$

Proof (1). Since Fl_n is a flower graph of order $2n + 1$. By definition,

$$\begin{aligned}
CD(Fl_n) &= \sum_{\{u,v\} \subseteq Fl_n} [\tau_{Fl_n}(u) + \tau_{Fl_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{Fl_n}(v_0) + \tau_{Fl_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{Fl_n}(v_0) + \tau_{Fl_n}(u_i)]d(v_0, u_i) + \sum_{\{u,v\} \subseteq U} [\tau_{Fl_n}(u) + \tau_{Fl_n}(v)]d(u,v) \\
&+ \sum_{\{u,v\} \subseteq V} [\tau_{Fl_n}(u) + \tau_{Fl_n}(v)]d(u,v) + \sum_{u \in U, v \in V} [\tau_{Fl_n}(u) + \tau_{Fl_n}(v)]d(u,v) \\
&= \sum_{i=1}^n (0 + n - 5)1 + \sum_{i=1}^n (0 + n - 2)1 + \sum_{\{u,v\} \subseteq U} (n - 2 + n - 2)d(u,v) + \sum_{\{u,v\} \subseteq V} (n - 5 + n - 5)d(u,v) \\
&+ \sum_{u \in U, v \in V} (n - 2 + n - 5)d(u,v) \\
&= n(n - 5) + n(n - 2) + 2(n - 2) \sum_{\{u,v\} \subseteq U} d(u,v) + 2(n - 5) \sum_{\{u,v\} \subseteq V} d(u,v) + (2n - 7) \sum_{u \in U, v \in V} d(u,v)
\end{aligned}$$

Using Lemma 2.6, we have

$$= 2n(4n^2 - 17n + 12).$$

(2).

$$\begin{aligned}
GC(Fl_n) &= \sum_{\{u,v\} \subseteq Fl_n} [\tau_{Fl_n}(u) \times \tau_{Fl_n}(v)]d(u,v) \\
&= \sum_{i=1}^n [\tau_{Fl_n}(v_0) \times \tau_{Fl_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{Fl_n}(v_0) \times \tau_{Fl_n}(u_i)]d(v_0, u_i) + \sum_{\{u,v\} \subseteq U} [\tau_{Fl_n}(u) \times \tau_{Fl_n}(v)]d(u,v) \\
&+ \sum_{\{u,v\} \subseteq V} [\tau_{Fl_n}(u) \times \tau_{Fl_n}(v)]d(u,v) + \sum_{u \in U, v \in V} [\tau_{Fl_n}(u) \times \tau_{Fl_n}(v)]d(u,v) \\
&= \sum_{i=1}^n (0)(n - 5)(1) + \sum_{i=1}^n (0)(n - 2)(1) + \sum_{\{u,v\} \subseteq U} (n - 2)^2 d(u,v) + \sum_{\{u,v\} \subseteq V} (n - 5)^2 d(u,v) \\
&+ \sum_{u \in U, v \in V} (n - 2)(n - 5)d(u,v) \\
&= 0 + 0 + (n - 2)^2 \sum_{\{u,v\} \subseteq U} d(u,v) + (n - 5)^2 \sum_{\{u,v\} \subseteq V} d(u,v) + (n - 2)(n - 5) \sum_{u \in U, v \in V} d(u,v) \\
&= 4n(n^3 - 8n^2 - 20n - 16).
\end{aligned}$$

Theorem 4.5. Let Sf_n be a sunflower graph of order $3n + 1$. Then, connection distance index and Gutman connection index for Sf_n are

$$(1) \quad CD(Sf_n) = 48n^3 - 55n^2 + 31n - 4,$$

$$(2) \quad GC(Sf_n) = 72n^4 - 156n^3 + 144n^2 - 56n + 2.$$

Proof (1). Since Sf_n is a flower graph of order $3n + 1$. By definition,

$$\begin{aligned} CD(Sf_n) &= \sum_{\{u,v\} \subseteq Sf_n} [\tau_{Sf_n}(u) + \tau_{Sf_n}(v)]d(u,v) \\ &= \sum_{i=1}^n [\tau_{v_n}(v_0) + \tau_{Sf_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{Sf_n}(v_0) + \tau_{Sf_n}(u_i)]d(v_0, u_i) + \sum_{i=1}^n [\tau_{Sf_n}(v_0) + \tau_{Sf_n}(w_i)]d(v_0, w_i) \\ &\quad + \sum_{\{u,v\} \subseteq U} [\tau_{Sf_n}(u) + \tau_{Sf_n}(v)]d(u,v) + \sum_{\{u,v\} \subseteq V} [\tau_{Sf_n}(u) + \tau_{Sf_n}(v)]d(u,v) + \sum_{\{u,v\} \subseteq W} [\tau_{Sf_n}(u) + \tau_{Sf_n}(v)]d(u,v) \\ &\quad + \sum_{u \in U, v \in V} [\tau_{Sf_n}(u) + \tau_{Sf_n}(v)]d(u,v) + \sum_{u \in U, v \in W} [\tau_{Sf_n}(u) + \tau_{Sf_n}(w)]d(u,v) + \sum_{u \in V, v \in W} [\tau_{Sf_n}(v) + \tau_{Sf_n}(w)]d(u,v) \\ &= \sum_{i=1}^n (0 + 3n - 4) + \sum_{i=1}^n (0 + 3n - 2)1 + \sum_{i=1}^n (0 + 3n - 1) + \sum_{\{u,v\} \subseteq U} (3n - 2 + 3n - 2)d(u,v) \\ &\quad + \sum_{\{u,v\} \subseteq V} (3n - 4 + 3n - 4)d(u,v) + \sum_{\{u,v\} \subseteq W} (3n - 1 + 3n - 1)d(u,v) + \sum_{u \in U, v \in V} (3n - 2 + 3n - 4)d(u,v) \\ &\quad + \sum_{u \in U, v \in W} (3n - 2 + 3n - 1)d(u,v) + \sum_{u \in V, v \in W} (3n - 4 + 3n - 1)d(u,v) \end{aligned}$$

Using Lemma 2.7, we have

$$= 48n^3 - 55n^2 + 31n - 4.$$

(2).

$$\begin{aligned} GC(Sf_n) &= \sum_{\{u,v\} \subseteq Sf_n} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(v)]d(u,v) \\ &= \sum_{i=1}^n [\tau_{v_n}(v_0) \times \tau_{Sf_n}(v_i)]d(v_0, v_i) + \sum_{i=1}^n [\tau_{Sf_n}(v_0) \times \tau_{Sf_n}(u_i)]d(v_0, u_i) + \sum_{i=1}^n [\tau_{Sf_n}(v_0) \times \tau_{Sf_n}(w_i)]d(v_0, w_i) \\ &\quad + \sum_{\{u,v\} \subseteq U} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(v)]d(u,v) + \sum_{\{u,v\} \subseteq V} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(v)]d(u,v) + \sum_{\{u,v\} \subseteq W} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(v)]d(u,v) \\ &\quad + \sum_{u \in U, v \in V} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(v)]d(u,v) + \sum_{u \in U, v \in W} [\tau_{Sf_n}(u) \times \tau_{Sf_n}(w)]d(u,v) + \sum_{u \in V, v \in W} [\tau_{Sf_n}(v) \times \tau_{Sf_n}(w)]d(u,v) \\ &= \sum_{i=1}^n (0)(3n - 4)(1) + \sum_{i=1}^n (0)(3n - 2)(1) + \sum_{i=1}^n (0)(3n - 1)(1) + \sum_{\{u,v\} \subseteq U} (3n - 2)^2 d(u,v) \\ &\quad + \sum_{\{u,v\} \subseteq V} (3n - 4)^2 d(u,v) + \sum_{\{u,v\} \subseteq W} (3n - 1)^2 d(u,v) + \sum_{u \in U, v \in V} (3n - 2)(3n - 4)d(u,v) \\ &\quad + \sum_{u \in U, v \in W} (3n - 2)(3n - 1)d(u,v) + \sum_{u \in V, v \in W} (3n - 4)(3n - 1)d(u,v) \\ &= 72n^4 - 156n^3 + 144n^2 - 56n + 2. \end{aligned}$$

5. Conclusions

The concluded remarks of this paper is as follows:

- we introduced several novel connection based Zagreb indices such as modified fourth Zagreb connection index (ZCI), multiplicative fourth ZCI, modified first multiplicative ZCI, modified second multiplicative ZCI, modified third multiplicative ZCI, connection distance index and Gutman connection index.
- We computed main results in the form of exact solutions for these newly defined connection based Zagreb indices of wheel-related graphs which are wheel graph, gear graph, helm graph, flower graph and sunflower graph respectively.
- We computed main results in the form of exact solutions for modified second ZCI, modified third ZCI, multiplicative first ZCI, multiplicative second ZCI and multiplicative third ZCI of above-mentioned wheel-related graphs.
- We also computed main results in the form of exact solutions for connection distance index and Gutman connection index of aforesaid special family of graphs.

Now, the problem is still open to compute novel connection based Zagreb indices for other family of graphs.

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