

The Generalized Zagreb Connection Indices of some Wheel Related Graphs

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Abstract

Topological indices (TIs) are the molecular descriptors that help us to predict physical and chemical properties of the molecules such as entropy, stability, enthalpy, boiling point, molecular weight, polarizability of organic compound, pressure, volume, density and infrared group frequency. These molecular descriptors mostly appear in the kind of degree based TIs. The first degree based TI was introduced by Gutman and Trinajstić in 1972 to find the total ϕ -electron energy of the concerned molecules. Such degree based TIs are further classified in two different ways; degree and connection number based TIs. The number of neighbors at distance 1 and 2 from vertex v of graph G are represented by degree and connection number, respectively. In this paper, we initially define connection number based TIs in the generalized versions with the name of generalized first Zagreb connection index (ZCI), generalized second ZCI, generalized third ZCI and generalized fourth ZCI. We also compute exact formulae for the abovementioned generalized versions of some wheel related graphs such as wheel, gear, helm, flower and sunflower. The obtained results ($C_{\alpha,\beta}^2, C_{\alpha,\beta}^3$ and $C_{\alpha,\beta}^4$) are the generalized extensions of the results (ZC_2^*, ZC_3^* and ZC_4^*) of Javaid et al. [Comp. J. Combin. Math. 2 (2020) 31-58] who worked only for $\alpha, \beta = 1$, respectively.

Keywords: Connection number; generalized Zagreb indices; ; wheel related graphs.

Mathematics Subject Classification: : 05C09, 05C92

1. Introduction

Chemical graph theory (CGT) is a subject which directly connects mathematical chemistry to graph theory and solves chemistry problems into mathematically vision, see [1]. In this field, we study several chemical structures with the help of topological indices (TIs). TIs are numeric quantities that gives information about the entire structure of graphs. They also useful in the study of physicochemical properties [2], the bioactivity of the chemical compounds [3] and to calculate the different types of organic compounds and also studied their chemical behaviors [4]. The quantitative structure property-relationship (QSPR) and quantitative structure activity-relationship (QSAR) are also investigated with the help of TIs and these relationships are beneficial for expensive experimental drug design, see [5, 6].

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The concept of degree was firstly introduced by Gutman and Trinajstić in [7] (1972) and they proved that the entire π -electron energy of a molecular graph depends on its structures. The mathematical formula of this graphical structure is known as first Zagreb index. In 1975, another descriptor was derived by Gutman et al. [8] with the name of second Zagreb index. After a long gap, Furtula and Gutman [9] (2015) worked on another TI with the name of third Zagreb index and investigated its various properties and called it by the name of F -index. At present, at least 5000 descriptors have been appeared in the field of CGT but abovementioned three Zagreb indices are more studied than others. For more properties and better applicability of Zagreb indices, we refer to [10, 11, 12, 13, 14, 15].

Azari [16] defined the generalized Zagreb index for ordinary and variable Zagreb indices and used this generalized version to find exact solutions of several graph products such as sum, corona, lexicographic, tensor, Cartesian, strong, disjunction and symmetric difference. Farahani and Kanna [17] used generalized Zagreb index to find the different structures of V-Phenylenic nanotubes and V-Phenylenic nanotori. Sarkar et al. [18] (2018) computed the generalized Zagreb index of graphene, carbon graphite and crystal cubic structure of carbons. Sarkar et al. used the generalized Zagreb index with another name as the (a,b)-Zagreb index of some regular dendrimers [19] and derived networks such as butterfly, mesh, Benes and Sierpinski [20]. Awais et al. [21] (2020) and Ali [22] (2021) defined some properties of metal-organic frameworks via the generalized and multiplicative versions of Zagreb indices. A few recent research on degree based TIs can be checked in this literatures [23, 24, 25, 26, 27, 28, 29, 30].

The concept of connection number to influence E_π was introduced by Gutman and Trinajstić in [7] (1972). This concept has started its work properly in 2017 [31] and 2018 [32]. Now, a list of researchers have put their eyes on the advancement of this concept. For that instance, Tang et al. [33] (2019) computed exact solutions for Zagreb connection indices (ZCIs) of subdivision-related operations on graphs. Ali et al. [34] enhanced the concept ZCIs in the shape of modified second and third ZCIs. Javaid et al. [35] introduced new modified version with the name of modified fourth ZCI. They also computed exact formulae for several newly defined ZCIs of wheel-related graphs. Shiladhar et al. [36] (2018) and Dayan et al. [37, 38] computed exact solutions for leap Zagreb indices, leap Gourava indices, leap reduced reciprocal Randic and leap reduced second Zagreb indices of some special graphs. In 2019, Kulli computed multiplicative leap Gourava indices [39] and (a,b)-Kulli-Basava indices [40] of some regular graphs. Recently, Javaid and Ali [41] computed exact expressions for the first general ZCI on some special graphs. Moreover, many articles have been published related to connection number based Zagreb indices, those can be seen here [42, 43, 44, 45, 46, 47, 48, 49].

In this paper, we define the generalized versions of the connection number based Zagreb indices such as the generalized first ZCI, generalized second ZCI, generalized third ZCI and generalized fourth ZCI of some wheel related graphs (wheel, gear, helm, flower and sunflower). The rest of the article is decorated as follows: Section II gives some notations and preliminary definitions of newly defined the generalized ZCIs and also discuss some important results, Section III provides the definitions of wheel related graphs and correspondence their main results of the generalized ZCIs and Section IV presents conclusion.

2. Notations and Preliminaries

Let G be a simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. The order ($|V(G)|$) and size ($|E(G)|$) in G are n and p . The length of the shortest path between any two vertices u and v is known as distance and it can be written as $d(u, v)$. In general, the open k -neighborhood of v is $N_k(v) = \{u \in V(G) : d(u, v) = k\}$. If d_k is the k -distance degree of a vertex v and it is also read as the number of k -neighbors of the vertex v then $d_k(v) = |N_k(v)|$. If we put $k = 1$ and 2 then we obtain degree ($d(v)$) and connection number ($d_2(v) = \tau(v)$) of a vertex v , respectively. If $e(v) = \max\{d(u, v) : u \in V(G)\}$, $D(G) = \max\{e(v) : v \in V(G)\}$ and $R(G) = \min\{e(v) : v \in V(G)\}$ are

the eccentricity, diameter and radius of v in G , respectively. For more information about terminology and notation, we suggest to see [50].

Definition 2.1 (See [21]). Let G be a connected graph. If $\forall \alpha, \beta \in \mathbb{R} - \{0\}$ and $k \geq 1$, then the k -distance generalized Zagreb index ($Z_{\alpha, \beta}^k(G)$) is as follows

$$Z_{\alpha, \beta}^k(G) = \sum_{uv \in E(G)} [d_k(u)^\alpha d_k(v)^\beta + d_k(v)^\alpha d_k(u)^\beta].$$

If $k = 1$, then $Z_{\alpha, \beta}(G)$ becomes degree of vertices based the generalized Zagreb index as

$$Z_{\alpha, \beta}(G) = \sum_{uv \in E(G)} [d(u)^\alpha d(v)^\beta + d(v)^\alpha d(u)^\beta].$$

The generalized Zagreb index [51] (2011) was introduced by Azari and Iranmanesh to find some networks of nanotubes and nanotubes. If we put $k = 2$, we can obtain connection number of vertices ($d_2 = \tau$) based Zagreb indices in the form of generalized Zagreb connection indices as follows:

Definition 2.2. Let G be a connected graph. If $\forall \alpha, \beta \in \mathbb{R} - \{0\}$, then the generalized first Zagreb connection index ($C_{\alpha, \beta}(G)$) is as follows

$$C_{\alpha, \beta}(G) = \sum_{uv \in E(G)} [\tau(u)^\alpha \tau(v)^\beta + \tau(v)^\alpha \tau(u)^\beta].$$

Definition 2.3. Let G be a connected graph. If $\forall \alpha, \beta \in \mathbb{R} - \{0\}$, then the generalized second Zagreb connection index ($C_{\alpha, \beta}^2(G)$) is as follows

$$C_{\alpha, \beta}^2(G) = \sum_{uv \in E(G)} [d(u)^\alpha \tau(v)^\beta + d(v)^\alpha \tau(u)^\beta].$$

Definition 2.4. Let G be a connected graph. If $\forall \alpha, \beta \in \mathbb{R} - \{0\}$, then the generalized third Zagreb connection index ($C_{\alpha, \beta}^3(G)$) is as follows

$$C_{\alpha, \beta}^3(G) = \sum_{uv \in E(G)} [d(u)^\alpha \tau(u)^\beta + d(v)^\alpha \tau(v)^\beta].$$

Definition 2.5. Let G be a connected graph. If $\forall \alpha, \beta \in \mathbb{R} - \{0\}$, then the generalized fourth Zagreb connection index ($C_{\alpha, \beta}^4(G)$) is as follows

$$C_{\alpha, \beta}^4(G) = \sum_{uv \in E(G)} [d(u)^\alpha \tau(u)^\beta \times d(v)^\alpha \tau(v)^\beta].$$

Now, we discuss some important results which may apply in the main results.

Lemma 2.1.[52] Let G be a connected graph with n vertices and p edges. Then, $\tau(u) + d(u) \leq \sum_{v \in N(u)} d(v)$, where equality holds if and only if G is a $\{C_3, C_4\}$ -free graph.

Lemma 2.2.[33] Let G be a connected and $\{C_3, C_4\}$ -free graph with n vertices and p edges. Then,

$$(i) \sum_{v \in V(G)} d(v) = 2p, \quad (ii) \sum_{v \in V(G)} \tau(v) = M_1(G) - 2p.$$

3. Definitions and Main Results

In this section, we define some wheel related graphs such as wheel, gear, helm, flower and sunflower, respectively. Then, we compute the generalized first, second, third and fourth Zagreb connection indices

for abovesaid graphs.

1. Wheel Graph W_n :

The wheel graph W_n is defined by joining K_1 and C_n . Where K_1 and C_n are the complete and cycle graphs of orders 1 and n respectively. So, order of wheel graph ($|V(W_n)|$) is $n + 1$ and size of wheel graph ($|E(W_n)|$) is $2n$. Apex is the vertex corresponding to K_1 and rim is the set of those vertices which is corresponding to C_n . The graphical depiction of wheel graph is presented in Figure 1.

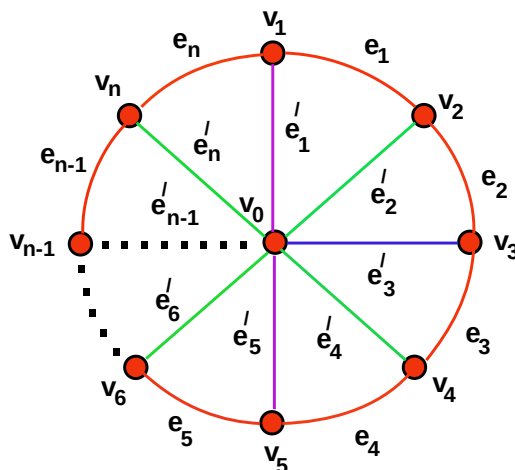


Figure 1. Wheel graph W_n

Table 1: Partition of vertices for W_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)
d_{W_n}	n	3
τ_{W_n}	0	n-3

Theorem 3.1. Let W_n be a wheel graph of order $n + 1$. Then, the generalized first, second, third and fourth Zagreb connection indices of W_n are

- (a) $C_{\alpha,\beta}(W_n) = 2n(n-3)^{\alpha+\beta}$,
- (b) $C_{\alpha,\beta}^2(W_n) = n(n-3)^\beta (n^\alpha + 2 \times 3^\alpha)$,
- (c) $C_{\alpha,\beta}^3(W_n) = 3^{\alpha+1} n(n-3)^\beta$,
- (d) $C_{\alpha,\beta}^4(W_n) = 3^{2\alpha} n(n-3)^{2\beta}$.

Proof (a). Since W_n is a wheel graph of order $n + 1$, where $n \geq 4$. Also, v_0 is the apex vertex and $\{v_1, v_2, \dots, v_n\}$ is the set of rim vertices for W_n . By definition,

$$\begin{aligned}
 C_{\alpha,\beta}(W_n) &= \sum_{uv \in E(W_n)} [\tau(u)^\alpha \tau(v)^\beta + \tau(v)^\alpha \tau(u)^\beta] \\
 &= \sum_{v_0 v_i \in E(W_n)} [\tau(v_0)^\alpha \tau(v_i)^\beta + \tau(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i v_{i+1} \in E(W_n)} [\tau(v_i)^\alpha \tau(v_{i+1})^\beta + \tau(v_{i+1})^\alpha \tau(v_i)^\beta] \\
 &= \sum_{i=1}^n [(0)^\alpha (n-3)^\beta + (n-3)^\alpha (0)^\beta] + \sum_{i=1}^n [(n-3)^\alpha (n-3)^\beta + (n-3)^\alpha (n-3)^\beta]
 \end{aligned}$$

$$= 2n(n-3)^{\alpha+\beta}$$

(b). By definition,

$$\begin{aligned} C_{\alpha,\beta}^2(W_n) &= \sum_{uv \in E(W_n)} [d(u)^\alpha \tau(v)^\beta + d(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(W_n)} [d(v_0)^\alpha \tau(v_i)^\beta + d(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i v_{i+1} \in E(W_n)} [d(v_i)^\alpha \tau(v_{i+1})^\beta + d(v_{i+1})^\alpha \tau(v_i)^\beta] \\ &= \sum_{i=1}^n [(n)^\alpha (n-3)^\beta + 3^\alpha (0)^\beta] + \sum_{i=1}^n [3^\alpha (n-3)^\beta + 3^\alpha (n-3)^\beta] \\ &= n(n-3)^\beta (n^\alpha + 2 \times 3^\alpha). \end{aligned}$$

(c). By definition,

$$\begin{aligned} C_{\alpha,\beta}^3(W_n) &= \sum_{uv \in E(W_n)} [d(u)^\alpha \tau(u)^\beta + d(v)^\alpha \tau(v)^\beta] \\ &= \sum_{v_0 v_i \in E(W_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i v_{i+1} \in E(W_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\ &= \sum_{i=1}^n [(n)^\alpha (0)^\beta + 3^\alpha (n-3)^\beta] + \sum_{i=1}^n [3^\alpha (n-3)^\beta + 3^\alpha (n-3)^\beta] \\ &= 3^{\alpha+1} n(n-3)^\beta. \end{aligned}$$

(d). By definition,

$$\begin{aligned} C_{\alpha,\beta}^4(W_n) &= \sum_{uv \in E(W_n)} [d(u)^\alpha \tau(u)^\beta \times d(v)^\alpha \tau(v)^\beta] \\ &= \sum_{v_0 v_i \in E(W_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i v_{i+1} \in E(W_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\ &= \sum_{i=1}^n [(n)^\alpha (0)^\beta \times 3^\alpha (n-3)^\beta] + \sum_{i=1}^n [3^\alpha (n-3)^\beta \times 3^\alpha (n-3)^\beta] \\ &= 3^{2\alpha} n(n-3)^{2\beta}. \end{aligned}$$

2. Gear Graph G_n :

The gear graph G_n is obtained from wheel graph W_n by including a new vertex between each pair of adjacent vertex of rim. So, order of gear graph ($|V(G_n)|$) is $2n+1$ and size of gear graph ($|E(G_n)|$) is $3n$. Also, the bipartite wheel graph is called gear graph. The graphical depiction of gear graph is presented in Figure 2.

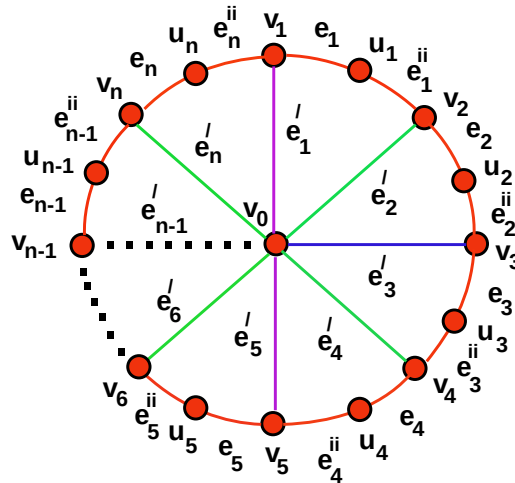


Figure 2. Gear graph G_n

Table 2: Partition of vertices for G_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Rim (u_i)
d_{G_n}	n	3	2
τ_{G_n}	n	n-1	3

Theorem 3.2. Let G_n be a gear graph of order $2n + 1$. Then, the generalized first, second, third and fourth Zagreb connection indices of G_n are

$$(a) \quad C_{\alpha, \beta}(G_n) = n(n-1)^\beta (n^\alpha + 2 \times 3^\alpha) + n(n-1)^\alpha (n^\beta + 2 \times 3^\beta),$$

$$(b) \quad C_{\alpha, \beta}^2(G_n) = n(n-1)^\beta (n^\alpha + 2^{\alpha+1}) + n3^\alpha (n^\beta + 2 \times 3^\beta),$$

$$(c) \quad C_{\alpha, \beta}^3(G_n) = n(n^{\alpha+\beta} + 2^{\alpha+1} 3^\beta) + 3^{\alpha+1} n(n-1)^\beta,$$

$$(d) \quad C_{\alpha, \beta}^4(G_n) = 3^\alpha n(n-1)^\beta (n^{\alpha+\beta} + 2^{\alpha+1} 3^\beta).$$

Proof (a). Since G_n is a gear graph of order $2n + 1$. Also, v_0 is the apex vertex, $\{v_1, v_2, \dots, v_n\}$, $\{u_1, u_2, \dots, u_n\}$ are the sets of rim vertices for G_n . By definition,

$$\begin{aligned} C_{\alpha, \beta}(G_n) &= \sum_{uv \in E(G_n)} [\tau(u)^\alpha \tau(v)^\beta + \tau(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(G_n)} [\tau(v_0)^\alpha \tau(v_i)^\beta + \tau(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i u_i \in E(G_n)} [\tau(v_i)^\alpha \tau(u_i)^\beta + \tau(u_i)^\alpha \tau(v_i)^\beta] \\ &= \sum_{i=1}^n [(n)^\alpha (n-1)^\beta + (n-1)^\alpha (n)^\beta] + \sum_{i=1}^{2n} [(n-1)^\alpha (3)^\beta + (3)^\alpha (n-1)^\beta] \\ &= n(n-1)^\beta (n^\alpha + 2 \times 3^\alpha) + n(n-1)^\alpha (n^\beta + 2 \times 3^\beta). \end{aligned}$$

(b). By definition,

$$\begin{aligned} C_{\alpha, \beta}^2(G_n) &= \sum_{uv \in E(G_n)} [d(u)^\alpha \tau(v)^\beta + d(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(G_n)} [d(v_0)^\alpha \tau(v_i)^\beta + d(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i u_i \in E(G_n)} [d(v_i)^\alpha \tau(u_i)^\beta + d(u_i)^\alpha \tau(v_i)^\beta] \end{aligned}$$

$$= \sum_{i=1}^n [(n)^\alpha (n-1)^\beta + (3)^\alpha (n)^\beta] + \sum_{i=1}^{2n} [(3)^\alpha (3)^\beta + (2)^\alpha (n-1)^\beta]$$

$$= n(n-1)^\beta (n^\alpha + 2^{\alpha+1}) + n3^\alpha (n^\beta + 2 \times 3^\beta).$$

(c). By definition,

$$C_{\alpha,\beta}^3(G_n) = \sum_{uv \in E(G_n)} [d(u)^\alpha \tau(u)^\beta + d(v)^\alpha \tau(v)^\beta]$$

$$= \sum_{v_0 v_i \in E(G_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i u_i \in E(G_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(u_i)^\alpha \tau(u_i)^\beta]$$

$$= \sum_{i=1}^n [(n)^\alpha (n)^\beta + (3)^\alpha (n-1)^\beta] + \sum_{i=1}^{2n} [(3)^\alpha (n-1)^\beta + (2)^\alpha (3)^\beta]$$

$$= n(n^{\alpha+\beta} + 2^{\alpha+1} 3^\beta) + 3^{\alpha+1} n(n-1)^\beta.$$

(d). By definition,

$$C_{\alpha,\beta}^4(G_n) = \sum_{uv \in E(G_n)} [d(u)^\alpha \tau(u)^\beta \times d(v)^\alpha \tau(v)^\beta]$$

$$= \sum_{v_0 v_i \in E(G_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i u_i \in E(G_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(u_i)^\alpha \tau(u_i)^\beta]$$

$$= \sum_{i=1}^n [(n)^\alpha (n)^\beta \times (3)^\alpha (n-1)^\beta] + \sum_{i=1}^{2n} [(3)^\alpha (n-1)^\beta \times (2)^\alpha (3)^\beta]$$

$$= 3^\alpha n(n-1)^\beta (n^{\alpha+\beta} + 2^{\alpha+1} 3^\beta).$$

3. Helm Graph H_n :

The helm graph H_n is obtained from wheel graph W_n by joining a pendant edge to the every vertex of the rim. So, order of helm graph ($|V(H_n)|$) is $2n + 1$ and size of helm graph ($|E(H_n)|$) is $3n$. The graphical depiction of helm graph is presented in Figure 3.

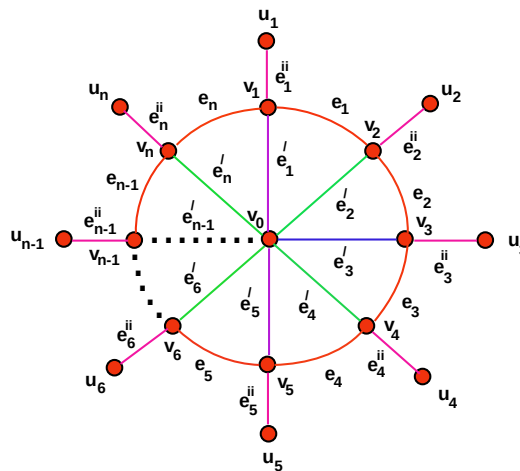


Figure 3. Helm graph H_n

Table 3: Partition of vertices for H_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Pendant (u_i)
d_{H_n}	n	4	1
τ_{H_n}	n	n-1	3

Theorem 3.3. Let H_n be a helm graph of order $2n + 1$. Then, the generalized first, second, third and fourth Zagreb connection indices of H_n are

$$(a) \quad C_{\alpha,\beta}(H_n) = n(n-1)^\beta(n^\alpha + 3^\alpha) + n(n-1)^\alpha[n^\beta + 3^\beta + 2(n-1)^\beta],$$

$$(b) \quad C_{\alpha,\beta}^2(H_n) = n(n-1)^\beta(n^\alpha + 1 + 2^{2\alpha+1}) + 4^\alpha n(n^\beta + 3^\beta),$$

$$(c) \quad C_{\alpha,\beta}^3(H_n) = n[n^{\alpha+\beta} + 4^{\alpha+1}(n-1)^\beta + 3^\beta],$$

$$(d) \quad C_{\alpha,\beta}^4(H_n) = 4^\alpha n(n-1)^\beta[n^{\alpha+\beta} + 4^\alpha(n-1)^\beta + 3^\beta].$$

Proof (a). Since H_n is a helm graph of order $2n + 1$. Also, v_0 is the apex vertex. $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_n\}$ are the sets of rim and pendant vertices for H_n . By definition,

$$\begin{aligned} C_{\alpha,\beta}(H_n) &= \sum_{uv \in E(H_n)} [\tau(u)^\alpha \tau(v)^\beta + \tau(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(H_n)} [\tau(v_0)^\alpha \tau(v_i)^\beta + \tau(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i u_i \in E(H_n)} [\tau(v_i)^\alpha \tau(u_i)^\beta + \tau(u_i)^\alpha \tau(v_i)^\beta] \\ &\quad + \sum_{v_i v_{i+1} \in E(H_n)} [\tau(v_i)^\alpha \tau(v_{i+1})^\beta + \tau(v_{i+1})^\alpha \tau(v_i)^\beta] \\ &= \sum_{i=1}^n [(n)^\alpha (n-1)^\beta + (n-1)^\alpha n^\beta] + \sum_{i=1}^n [(n-1)^\alpha 3^\beta + 3^\alpha (n-1)^\beta] + \sum_{i=1}^n [(n-1)^\alpha (n-1)^\beta + (n-1)^\alpha (n-1)^\beta] \\ &= n(n-1)^\beta(n^\alpha + 3^\alpha) + n(n-1)^\alpha[n^\beta + 3^\beta + 2(n-1)^\beta]. \end{aligned}$$

(b). By definition,

$$\begin{aligned} C_{\alpha,\beta}^2(H_n) &= \sum_{uv \in E(H_n)} [d(u)^\alpha \tau(v)^\beta + d(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(H_n)} [d(v_0)^\alpha \tau(v_i)^\beta + d(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i u_i \in E(H_n)} [d(v_i)^\alpha \tau(u_i)^\beta + d(u_i)^\alpha \tau(v_i)^\beta] \\ &\quad + \sum_{v_i v_{i+1} \in E(H_n)} [d(v_i)^\alpha \tau(v_{i+1})^\beta + d(v_{i+1})^\alpha \tau(v_i)^\beta] \\ &= \sum_{i=1}^n [(n)^\alpha (n-1)^\beta + (4)^\alpha (n)^\beta] + \sum_{i=1}^n [(4)^\alpha (3)^\beta + (1)^\alpha (n-1)^\beta] + \sum_{i=1}^n [(4)^\alpha (n-1)^\beta + (4)^\alpha (n-1)^\beta] \\ &= n(n-1)^\beta(n^\alpha + 1 + 2^{2\alpha+1}) + 4^\alpha n(n^\beta + 3^\beta). \end{aligned}$$

(c). By definition,

$$\begin{aligned} C_{\alpha,\beta}^3(H_n) &= \sum_{uv \in E(H_n)} [d(u)^\alpha \tau(u)^\beta + d(v)^\alpha \tau(v)^\beta] \\ &= \sum_{v_0 v_i \in E(H_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i u_i \in E(H_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(u_i)^\alpha \tau(u_i)^\beta] \end{aligned}$$

$$\begin{aligned}
& + \sum_{v_i v_{i+1} \in E(H_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\
& = \sum_{i=1}^n [(n)^\alpha (n)^\beta + (4)^\alpha (n-1)^\beta] + \sum_{i=1}^n [(4)^\alpha (n-1)^\beta + (1)^\alpha (3)^\beta] + \sum_{i=1}^n [(4)^\alpha (n-1)^\beta + (4)^\alpha (n-1)^\beta] \\
& = n[n^{\alpha+\beta} + 4^{\alpha+1}(n-1)^\beta + 3^\beta].
\end{aligned}$$

(d). By definition,

$$\begin{aligned}
C_{\alpha,\beta}^4(H_n) &= \sum_{uv \in E(H_n)} [d(u)^\alpha \tau(u)^\beta \times d(v)^\alpha \tau(v)^\beta] \\
&= \sum_{v_0 v_i \in E(H_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i u_i \in E(H_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(u_i)^\alpha \tau(u_i)^\beta] \\
&+ \sum_{v_i v_{i+1} \in E(H_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\
&= \sum_{i=1}^n [(n)^\alpha (n)^\beta \times (4)^\alpha (n-1)^\beta] + \sum_{i=1}^n [(4)^\alpha (n-1)^\beta \times (1)^\alpha (3)^\beta] + \sum_{i=1}^n [(4)^\alpha (n-1)^\beta \times (4)^\alpha (n-1)^\beta] \\
&= 4^\alpha n(n-1)^\beta [n^{\alpha+\beta} + 4^\alpha (n-1)^\beta + 3^\beta].
\end{aligned}$$

4. Flower Graph Fl_n :

The flower graph Fl_n is obtained from helm graph H_n by joining each pendant vertex to the apex of helm graph. So, order of flower graph ($|V(Fl_n)|$) is $2n+1$ and size of flower graph ($|E(Fl_n)|$) is $4n$. The graphical depiction of flower graph is presented in Figure 4.

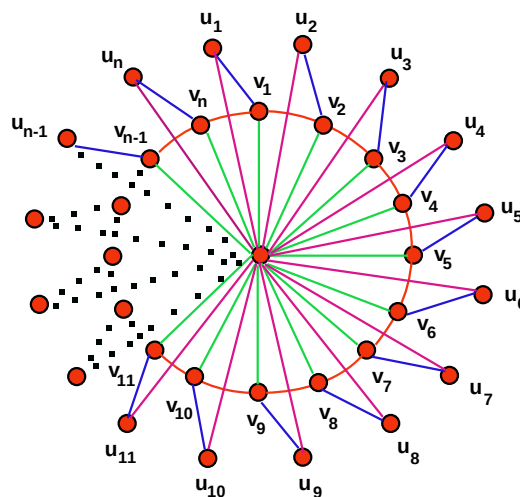


Figure 4. Flower graph Fl_n

Table 4: Partition of vertices for Fl_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Extreme (u_i)
d_{Fl_n}	2n	4	2
τ_{Fl_n}	0	2n-4	2n-2

Theorem 3.4. Let Fl_n be a flower graph of order $2n + 1$. Then, the generalized first, second, third and fourth Zagreb connection indices of Fl_n are

$$(a) \quad C_{\alpha,\beta}(Fl_n) = n(2n-4)^\alpha [(2n-2)^\beta + 2(2n-4)^\beta] + n(2n-2)^\alpha (2n-4)^\beta,$$

$$(b) \quad C_{\alpha,\beta}^2(Fl_n) = 2^\alpha n(2n-4)^\beta (n^\alpha + 2^{\alpha+1} + 1) + 2^\alpha n(2n-2)^\beta (n^\alpha + 2^\alpha),$$

$$(c) \quad C_{\alpha,\beta}^3(Fl_n) = 2^{\alpha+1} n[2^{\alpha+1} (2n-4)^\beta + (2n-2)^\beta],$$

$$(d) \quad C_{\alpha,\beta}^4(Fl_n) = 2^{3\alpha} n(2n-4)^\beta [(2n-2)^\beta + 2^\alpha (2n-4)^\beta].$$

Proof (a). Since Fl_n is a flower graph of order $2n + 1$. Also, v_0 is the apex vertex. $\{v_1, v_2, \dots, v_n\}$ and $\{u_1, u_2, \dots, u_n\}$ are the sets of rim and extreme vertices for Fl_n . By definition,

$$\begin{aligned} C_{\alpha,\beta}(Fl_n) &= \sum_{uv \in E(Fl_n)} [\tau(u)^\alpha \tau(v)^\beta + \tau(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(Fl_n)} [\tau(v_0)^\alpha \tau(v_i)^\beta + \tau(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_0 u_i \in E(Fl_n)} [\tau(v_0)^\alpha \tau(u_i)^\beta + \tau(u_i)^\alpha \tau(v_0)^\beta] \\ &+ \sum_{v_i u_i \in E(Fl_n)} [\tau(v_i)^\alpha \tau(u_i)^\beta + \tau(u_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i v_{i+1} \in E(Fl_n)} [\tau(v_i)^\alpha \tau(v_{i+1})^\beta + \tau(v_{i+1})^\alpha \tau(v_i)^\beta] \\ &= \sum_{i=1}^n [(0)^\alpha (2n-4)^\beta + (2n-4)^\alpha (0)^\beta] + \sum_{i=1}^n [(0)^\alpha (2n-2)^\beta + (2n-2)^\alpha (0)^\beta] + \sum_{i=1}^n [(2n-4)^\alpha (2n-2)^\beta \\ &+ (2n-2)^\alpha (2n-4)^\beta] + \sum_{i=1}^n [(2n-4)^\alpha (2n-4)^\beta + (2n-4)^\alpha (2n-4)^\beta] \\ &= n(2n-4)^\alpha [(2n-2)^\beta + 2(2n-4)^\beta] + n(2n-2)^\alpha (2n-4)^\beta. \end{aligned}$$

(b). By definition,

$$\begin{aligned} C_{\alpha,\beta}^2(Fl_n) &= \sum_{uv \in E(Fl_n)} [d(u)^\alpha \tau(v)^\beta + d(v)^\alpha \tau(u)^\beta] \\ &= \sum_{v_0 v_i \in E(Fl_n)} [d(v_0)^\alpha \tau(v_i)^\beta + d(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_0 u_i \in E(Fl_n)} [d(v_0)^\alpha \tau(u_i)^\beta + d(u_i)^\alpha \tau(v_0)^\beta] \\ &+ \sum_{v_i u_i \in E(Fl_n)} [d(v_i)^\alpha \tau(u_i)^\beta + d(u_i)^\alpha \tau(v_i)^\beta] + \sum_{v_i v_{i+1} \in E(Fl_n)} [d(v_i)^\alpha \tau(v_{i+1})^\beta + d(v_{i+1})^\alpha \tau(v_i)^\beta] \\ &= \sum_{i=1}^n [(2n)^\alpha (2n-4)^\beta + (4)^\alpha (0)^\beta] + \sum_{i=1}^n [(2n)^\alpha (2n-2)^\beta + (2)^\alpha (0)^\beta] + \sum_{i=1}^n [(4)^\alpha (2n-2)^\beta \\ &+ (2)^\alpha (2n-4)^\beta] + \sum_{i=1}^n [(4)^\alpha (2n-4)^\beta + (4)^\alpha (2n-4)^\beta] \\ &= 2^\alpha n(2n-4)^\beta (n^\alpha + 2^{\alpha+1} + 1) + 2^\alpha n(2n-2)^\beta (n^\alpha + 2^\alpha). \end{aligned}$$

(c). By definition,

$$\begin{aligned} C_{\alpha,\beta}^3(Fl_n) &= \sum_{uv \in E(Fl_n)} [d(u)^\alpha \tau(u)^\beta + d(v)^\alpha \tau(v)^\beta] \\ &= \sum_{v_0 v_i \in E(Fl_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_0 u_i \in E(Fl_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(u_i)^\alpha \tau(u_i)^\beta] \end{aligned}$$

$$\begin{aligned}
& + \sum_{v_i u_i \in E(Fl_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(u_i)^\alpha \tau(u_i)^\beta] + \sum_{v_i v_{i+1} \in E(Fl_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\
& = \sum_{i=1}^n [(2n)^\alpha (0)^\beta + (4)^\alpha (2n-4)^\beta] + \sum_{i=1}^n [(2n)^\alpha (0)^\beta + (2)^\alpha (2n-2)^\beta] + \sum_{i=1}^n [(4)^\alpha (2n-4)^\beta \\
& + (2)^\alpha (2n-2)^\beta] + \sum_{i=1}^n [(4)^\alpha (2n-4)^\beta + (4)^\alpha (2n-4)^\beta] \\
& = 2^{\alpha+1} n [2^{\alpha+1} (2n-4)^\beta + (2n-2)^\beta].
\end{aligned}$$

(d). By definition,

$$\begin{aligned}
C_{\alpha,\beta}^4(Fl_n) &= \sum_{uv \in E(Fl_n)} [d(u)^\alpha \tau(u)^\beta \times d(v)^\alpha \tau(v)^\beta] \\
&= \sum_{v_0 v_i \in E(Fl_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_0 u_i \in E(Fl_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(u_i)^\alpha \tau(u_i)^\beta] \\
&+ \sum_{v_i u_i \in E(Fl_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(u_i)^\alpha \tau(u_i)^\beta] + \sum_{v_i v_{i+1} \in E(Fl_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\
&= \sum_{i=1}^n [(2n)^\alpha (0)^\beta \times (4)^\alpha (2n-4)^\beta] + \sum_{i=1}^n [(2n)^\alpha (0)^\beta \times (2)^\alpha (2n-2)^\beta] + \sum_{i=1}^n [(4)^\alpha (2n-4)^\beta \\
&\times (2)^\alpha (2n-2)^\beta] + \sum_{i=1}^n [(4)^\alpha (2n-4)^\beta \times (4)^\alpha (2n-4)^\beta] \\
&= 2^{3\alpha} n (2n-4)^\beta [(2n-2)^\beta + 2^\alpha (2n-4)^\beta].
\end{aligned}$$

5. Sunflower Graph Sf_n :

The sunflower graph Sf_n is obtained from flower graph Fl_n by growing n pendant edges to the apex of flower graph. So, order of sunflower graph ($|V(Sf_n)|$) is $3n + 1$ and size of sunflower graph ($|E(Sf_n)|$) is $5n$. The graphical depiction of sunflower graph is presented in Figure 5.

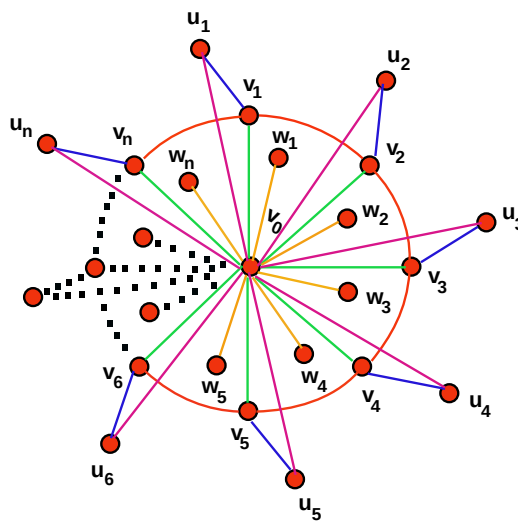


Figure 5. Sunflower graph Sf_n

Table 5: Partition of vertices for Sf_n .

Degree/Connection number	Apex (v_0)	Rim (v_i)	Extreme (u_i)	Pendant (w_i)
d_{Sf_n}	$3n$	4	2	1
τ_{Sf_n}	0	$3n-4$	$3n-2$	$3n-1$

Theorem 3.5. Let Sf_n be a sunflower graph of order $3n + 1$. Then, the generalized first, second, third and fourth Zagreb connection indices of Sf_n are

- (a) $C_{\alpha,\beta}(Sf_n) = n(3n-4)^\alpha[(3n-2)^\beta + 2(3n-4)^\beta] + n(3n-2)^\alpha(3n-4)^\beta$,
- (b) $C_{\alpha,\beta}^2(Sf_n) = n(3n-4)^\beta[3^\alpha n^\alpha + 2^\alpha(1+2^{\alpha+1})] + n(3n-2)^\beta(3^\alpha n^\alpha + 4^\alpha) + 3^\alpha n^{\alpha+1}(3n-1)^\beta$,
- (c) $C_{\alpha,\beta}^3(Sf_n) = n[4^{\alpha+1}(3n-4)^\beta + 2^{\alpha+1}(3n-2)^\beta + (3n-1)^\beta]$,
- (d) $C_{\alpha,\beta}^4(Sf_n) = 4^\alpha n(3n-4)^\beta[2^\alpha(3n-2)^\beta + 4^\alpha(3n-4)^\beta]$.

Proof (a). Since Sf_n is a sunflower graph of order $3n + 1$. Also, v_0 is the apex vertex. $\{v_1, v_2, \dots, v_n\}$, $\{u_1, u_2, \dots, u_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are the sets of rim, extreme and pendant vertices for Sf_n . By definition,

$$\begin{aligned}
 C_{\alpha,\beta}(Sf_n) &= \sum_{uv \in E(Sf_n)} [\tau(u)^\alpha \tau(v)^\beta + \tau(v)^\alpha \tau(u)^\beta] \\
 &= \sum_{v_0 v_i \in E(Sf_n)} [\tau(v_0)^\alpha \tau(v_i)^\beta + \tau(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_0 u_i \in E(Sf_n)} [\tau(v_0)^\alpha \tau(u_i)^\beta + \tau(u_i)^\alpha \tau(v_0)^\beta] \\
 &+ \sum_{v_0 w_i \in E(Sf_n)} [\tau(v_0)^\alpha \tau(w_i)^\beta + \tau(w_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i u_i \in E(Sf_n)} [\tau(v_i)^\alpha \tau(u_i)^\beta + \tau(u_i)^\alpha \tau(v_i)^\beta] \\
 &+ \sum_{v_i v_{i+1} \in E(Sf_n)} [\tau(v_i)^\alpha \tau(v_{i+1})^\beta + \tau(v_{i+1})^\alpha \tau(v_i)^\beta] \\
 &= \sum_{i=1}^n [(0)^\alpha (3n-4)^\beta + (3n-4)^\alpha (0)^\beta] + \sum_{i=1}^n [(0)^\alpha (3n-2)^\beta + (3n-2)^\alpha (0)^\beta] + \sum_{i=1}^n [(0)^\alpha (3n-1)^\beta \\
 &+ (3n-1)^\alpha (0)^\beta] + \sum_{i=1}^n [(3n-4)^\alpha (3n-2)^\beta + (3n-2)^\alpha (3n-4)^\beta] + \sum_{i=1}^n [(3n-4)^\alpha (3n-4)^\beta \\
 &+ (3n-4)^\alpha (3n-4)^\beta] \\
 &= n(3n-4)^\alpha[(3n-2)^\beta + 2(3n-4)^\beta] + n(3n-2)^\alpha(3n-4)^\beta.
 \end{aligned}$$

(b). By definition,

$$\begin{aligned}
 C_{\alpha,\beta}^2(Sf_n) &= \sum_{uv \in E(Sf_n)} [d(u)^\alpha \tau(v)^\beta + d(v)^\alpha \tau(u)^\beta] \\
 &= \sum_{v_0 v_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_i)^\beta + d(v_i)^\alpha \tau(v_0)^\beta] + \sum_{v_0 u_i \in E(Sf_n)} [d(v_0)^\alpha \tau(u_i)^\beta + d(u_i)^\alpha \tau(v_0)^\beta] \\
 &+ \sum_{v_0 w_i \in E(Sf_n)} [d(v_0)^\alpha \tau(w_i)^\beta + d(w_i)^\alpha \tau(v_0)^\beta] + \sum_{v_i u_i \in E(Sf_n)} [d(v_i)^\alpha \tau(u_i)^\beta + d(u_i)^\alpha \tau(v_i)^\beta] \\
 &+ \sum_{v_i v_{i+1} \in E(Sf_n)} [d(v_i)^\alpha \tau(v_{i+1})^\beta + d(v_{i+1})^\alpha \tau(v_i)^\beta]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n [(3n)^\alpha (3n-4)^\beta + (4)^\alpha (0)^\beta] + \sum_{i=1}^n [(3n)^\alpha (3n-2)^\beta + (2)^\alpha (0)^\beta] + \sum_{i=1}^n [(3n)^\alpha (3n-1)^\beta \\
 &+ (1)^\alpha (0)^\beta] + \sum_{i=1}^n [(4)^\alpha (3n-2)^\beta + (2)^\alpha (3n-4)^\beta] + \sum_{i=1}^n [(4)^\alpha (3n-4)^\beta \\
 &+ (4)^\alpha (3n-4)^\beta] \\
 &= n(3n-4)^\beta [3^\alpha n^\alpha + 2^\alpha (1+2^{\alpha+1})] + n(3n-2)^\beta (3^\alpha n^\alpha + 4^\alpha) + 3^\alpha n^{\alpha+1} (3n-1)^\beta.
 \end{aligned}$$

(c). By definition,

$$\begin{aligned}
 C_{\alpha,\beta}^3(Sf_n) &= \sum_{uv \in E(Sf_n)} [d(u)^\alpha \tau(u)^\beta + d(v)^\alpha \tau(v)^\beta] \\
 &= \sum_{v_0 v_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_0 u_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(u_i)^\alpha \tau(u_i)^\beta] \\
 &+ \sum_{v_0 w_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_0)^\beta + d(w_i)^\alpha \tau(w_i)^\beta] + \sum_{v_i u_i \in E(Sf_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(u_i)^\alpha \tau(u_i)^\beta] \\
 &+ \sum_{v_i v_{i+1} \in E(Sf_n)} [d(v_i)^\alpha \tau(v_i)^\beta + d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\
 &= \sum_{i=1}^n [(3n)^\alpha (0)^\beta + (4)^\alpha (3n-4)^\beta] + \sum_{i=1}^n [(3n)^\alpha (0)^\beta + (2)^\alpha (3n-2)^\beta] + \sum_{i=1}^n [(3n)^\alpha (0)^\beta + (1)^\alpha (3n-1)^\beta] \\
 &+ \sum_{i=1}^n [(4)^\alpha (3n-4)^\beta + (2)^\alpha (3n-2)^\beta] + \sum_{i=1}^n [(4)^\alpha (3n-4)^\beta + (4)^\alpha (3n-4)^\beta] \\
 &= n[4^{\alpha+1} (3n-4)^\beta + 2^{\alpha+1} (3n-2)^\beta + (3n-1)^\beta].
 \end{aligned}$$

(d). By definition,

$$\begin{aligned}
 C_{\alpha,\beta}^4(Sf_n) &= \sum_{uv \in E(Sf_n)} [d(u)^\alpha \tau(u)^\beta \times d(v)^\alpha \tau(v)^\beta] \\
 &= \sum_{v_0 v_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(v_i)^\alpha \tau(v_i)^\beta] + \sum_{v_0 u_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(u_i)^\alpha \tau(u_i)^\beta] \\
 &+ \sum_{v_0 w_i \in E(Sf_n)} [d(v_0)^\alpha \tau(v_0)^\beta \times d(w_i)^\alpha \tau(w_i)^\beta] + \sum_{v_i u_i \in E(Sf_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(u_i)^\alpha \tau(u_i)^\beta] \\
 &+ \sum_{v_i v_{i+1} \in E(Sf_n)} [d(v_i)^\alpha \tau(v_i)^\beta \times d(v_{i+1})^\alpha \tau(v_{i+1})^\beta] \\
 &= \sum_{i=1}^n [(3n)^\alpha (0)^\beta \times (4)^\alpha (3n-4)^\beta] + \sum_{i=1}^n [(3n)^\alpha (0)^\beta \times (2)^\alpha (3n-2)^\beta] + \sum_{i=1}^n [(3n)^\alpha (0)^\beta \times (1)^\alpha (3n-1)^\beta] \\
 &+ \sum_{i=1}^n [(4)^\alpha (3n-4)^\beta \times (2)^\alpha (3n-2)^\beta] + \sum_{i=1}^n [(4)^\alpha (3n-4)^\beta \times (4)^\alpha (3n-4)^\beta] \\
 &= 4^\alpha n (3n-4)^\beta [2^\alpha (3n-2)^\beta + 4^\alpha (3n-4)^\beta].
 \end{aligned}$$

4. Conclusion

In this paper, we computed the generalized Zagreb connection indices (ZCIs) such as the generalized first ZCI, generalized second ZCI, generalized third ZCI and generalized fourth ZCI of some wheel related graphs (wheel, gear, helm, flower and sunflower).

Now, the problem is still open to compute the generalized ZCIs for windmill graph, Dutch windmill graph and the line graph of Dutch windmill graph.

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References

- [1] N. Trinajstić, Chemical Graph Theory, CRC Press, Boca Raton, FL, USA, 1992.
- [2] M. S. Husin, R. Hasni, N. E. Arif and M. Imran, On topological indices of certain families of nanostar dendrimers, *Molecules*, 21(7)(2016), 10 pages.
- [3] W. Gao, H. Wu, M. K. Siddiqui and A. Q. Baig, Study of biological networks using graph theory, *Saudi Journal of Biological Sciences*, 25(2018), 1212–1219.
- [4] A. R. Matamala and E. Estrada, Generalised topological indices: optimisation methodology and physico-chemical interpretation, *Chemical Physics Letters*, 410(4-6)(2005), 343–347.
- [5] R. Gozalbes, J. P. Doucet and F. Derouin, Application of topological descriptors in QSAR and drug design: history and new trends, *Current Drug Targets-Infectious Disorders*, 2(1)(2002), 93–102.
- [6] F. Yan, Q. Shang, S. Xia, Q. Wang and P. Ma, Application of topological index in predicting ionic liquids densities by the quantitative structure property relationship method, *Journal of Chemical Engineering Data*, 60(3)(2015), 734–739.
- [7] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. III. total π -electron energy of alternant hydrocarbons, *Chemical Physics Letters*, 17(1972), 535–538.
- [8] I. Gutman, B. Ruscic, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals. XII. acyclic polyenes, *Journal of Chemical Physics*, 62(9)(1975), 3399–3405.
- [9] B. Furtula and I. Gutman, A forgotten topological index, *Journal of Mathematical Chemistry*, 53(4)(2015), 1184–1190.
- [10] I. Gutman and O. Polansky, Mathematical concepts in organic chemistry, Springer-Verlag, Berlin, Germany, 1986.
- [11] K. C. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Communications in Mathematical and in Computer Chemistry*, 52(2004), 103–112.
- [12] I. Gutman and K. C. Das, The first Zagreb index 30 years after, *MATCH Communications in Mathematical and in Computer Chemistry*, 50(2004), 83–92.
- [13] B. Liu and I. Gutman, Estimating the Zagreb and the general randic indices, *MATCH Communications in Mathematical and in Computer Chemistry*, 57(2007), 617–632.
- [14] M. Liu and B. Liu, Some properties of the first general Zagreb index, *Australasian Journal of Combinatorics*, 47(2010), 285–294.
- [15] I. Gutman, Degree-based topological indices, *Croatica Chemica Acta*, 86(4)(2013), 351–361.
- [16] M. AZARI, Generalized Zagreb index of product graphs, *Transactions on Combinatorics*, 8(4)(2019), 35–48.
- [17] M. R. Farahani and M. R. R. Kanna, Generalized Zagreb index of V-phenylenic nanotubes and nanotori, *Journal of Chemical and Pharmaceutical Research*, 7(11)(2015), 241–245.
- [18] P. Sarkar, N. De and A. Pala, The generalized Zagreb index of some carbon structures, *Acta Chemica Iasi*, 26(1)(2018), 91–104.
- [19] P. Sarkar, N. De, I. N. Cangul, and A. Pal, Generalized Zagreb index of some dendrimer structures, *Universal Journal of Mathematics and Applications*, 1(3)(2018), 160–165.
- [20] P. Sarkar, N. De, I. N. Cangul, and A. Pal, The (a,b)-Zagreb index of some derived networks, *Journal of Taibah University for Science*, 13(1)(2019), 79–86.
- [21] H. M. Awais, M. Jamal and M. Javaid, Topological properties of metal-organic frameworks, *Main Group Metal Chemistry*, vol. 43, pp. 67–76, 2020.
- [22] U. Ali, Computing novel multiplicative Zagreb connection indices of Metal-Organic Networks (MONs), *Scientific Inquiry and Review*, 5(1)(2021), 46–71.
- [23] J.-B. Liu, S. Javed, M. Javaid and K. Shabbir, Computing first general Zagreb index of operations on graphs, *IEEE Access*, 7(2019), 47494–47502.
- [24] H. M. Awais, M. Javaid and A. Ali, First general Zagreb index of generalized F-sum graphs, *Discrete Dynamics in Nature and Society*, 2020(2020), ID 2954975, 16 pages.
- [25] B. Chaluvvaraju, H. S. Boregowda and I. N. Cangul, Some inequalities for the first general Zagreb index of graphs n and line graphs, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* (2020), <https://doi.org/10.1007/s40010-020-00679-9>.
- [26] Y. Shang, Vulnerability of networks: fractional percolation on random graphs, *Physical Review E*, 28(2014), ID phyrevE.89.012813.
- [27] J.-B. Liu, B. Ali, M. A. Malik, H. M. A. Siddiqui and M. Imran, Reformulated Zagreb indices of some derived graphs, *Mathematics*, 7(2019), ID math7040366, 14 pages.
- [28] I. Gutman, B. Furtula, Z. K. Vukićević and G. Popivoda, On Zagreb indices and coindices, *MATCH Communications in Mathematical and in Computer Chemistry*, 74 (2015), 5–16.
- [29] W. Gao, M. K. Siddiqui, M. Naeem and M. Imran, Computing multiplie *ABC* index and multiple *GA* index of some grid grpahs, *Open Physics*, 16(2018), 588–598.
- [30] S. Rose and S. Naduvath, On chromatic topological indices of certain wheel related graphs, *International Journal of Mathematics, Science and Engineering Applications*, 2(3)(2017), 129–144.
- [31] A. M. Naji, N. D. Soner and I. Gutman, On leap Zagreb indices of graphs, *Communications in Combinatorics and Optimization*, 2(2)(2017), 99–117.
- [32] A. Ali and N. Trinajstić, A novel/old modification of the first Zagreb index, *Molecular Informatics*, 37(2018), 1–7.
- [33] J.-H. Tang, U. Ali, M. Javaid, and K. Shabbir, Zagreb connection indices of subdivision and semi-total point operations on graphs, *Journal of Chemistry*, 2019(2019), Article ID 9846913, 14 pages.
- [34] U. Ali, M. Javaid and A. Kashif, Modified Zagreb connection indices of the T-sum graphs, *Main Group Metal Chemistry*, 43(1)(2020), 43–55.
- [35] M. Javaid, M. K. Siddiqui and U. Ali, Novel connection based Zagreb indices of several wheel-related graphs, *Computational Journal of Combinatorial Mathematics*, 2(2020), 31–58.
- [36] P. Shiladhar, A. M. Naji and N. D. Soner, Leap Zagreb indices of some wheel related graphs, *Journal of Computer and Mathematical Sciences*, 9(3)(2018), 221–231.
- [37] F. Dayan, M. Javaid and M. A. ur Rehman, On leap Gourava indices of some wheel related graphs, *Scientific Inquiry and Review*, 2(4)(2018), 14–24.
- [38] F. Dayan, M. Javaid and M. A. Rehman, On leap reduced reciprocal Randić and leap reduced second Zagreb indices of some graphs, *Scientific Inquiry and Review*, 3(2)(2019), 27–35.
- [39] V. R. Kulli, On multiplicative leap Gourava indices of graphs, *International Journal of Engineering Sciences and Research Technology*, 8(10)(2019), 22–30.
- [40] V. R. Kulli, The (a, b)-Kulli-Basava index of graphs, *International Journal of Engineering Sciences and Research Technology*, 8(12)(2019), 103–112.
- [41] M. Javaid and U. Ali, The first general Zagreb connection index and its particular cases on some special graphs, *Computational Journal of Combinatorial Mathematics*, 1(2021), 1–10.

- [42] V. R. Kulli, Minus leap and square leap indices and their polynomials of some special graphs, *International Research Journal of pure Algebra*, 8(11)(2018), 54-60.
- [43] U. Ali, and M. Javaid, Zagreb connection indices of disjunction and symmetric difference operations on graphs, *Journal of Prime Research in Mathematics*, 16(2)(2020), 1-15.
- [44] U. Ali, and M. Javaid, Upper bounds of Zagreb connection indices of tensor and strong product on graphs, *Punjab University Journal of Mathematics*, 52(4)(2020), 89-100.
- [45] J. Cao, U. Ali, M. Javaid and C. Huang, Zagreb connection indices of molecular graphs based on operations, *Complexity*, 2020(2020), Article ID 7385682, 15 pages.
- [46] J.-B. Liu, Z. Raza, and M. Javaid, Zagreb connection numbers for cellular neural networks, *Discrete Dynamics in Nature and Society*, 2020(2020), Article ID 8038304, 8 pages.
- [47] M. Javaid, U. Ali, and J.-B. Liu, Computing analysis for first Zagreb connection index and coindex of resultant graphs, *Mathematical Problems in Engineering*, 2021(2021), Article ID 6019517, 19 pages.
- [48] U. Ali, M. Javaid, and A. M. Alanazi, Computing analysis of connection-based indices and coindices for product of molecular networks, *Symmetry*, 12(8)(2020), 12081320, 23 pages.
- [49] M. T. Hussain, M. Javaid, U. Ali, A. Raza and M. N. Alam, Comparing zinc oxide- and zinc silicate-related metal-organic networks via connection-based Zagreb indices, *Journal of Chemistry*, vol. 2021, Article ID 5066394, 16 pages, 2021.
- [50] F. Harary, *Graph theory*, Addison-Wesley Publishing Co., Reading Mass. Menlo Park, Calif. London, 1969.
- [51] M. Azari, A. Iranmanesh, *Generalized Zagreb index of graphs*, *Studia Univ. Babes Bolyai Chem.*, 56(2011), 59-70.
- [52] S. Yamaguchi, Estimating the Zagreb indices and the spectral radius of triangle- and quadrangle-free connected graphs, *Chem. Phys. Lett.* 458(4)(2008), 396-398.